Preliminary Mathematics for online MSc programmes in Data Analytics

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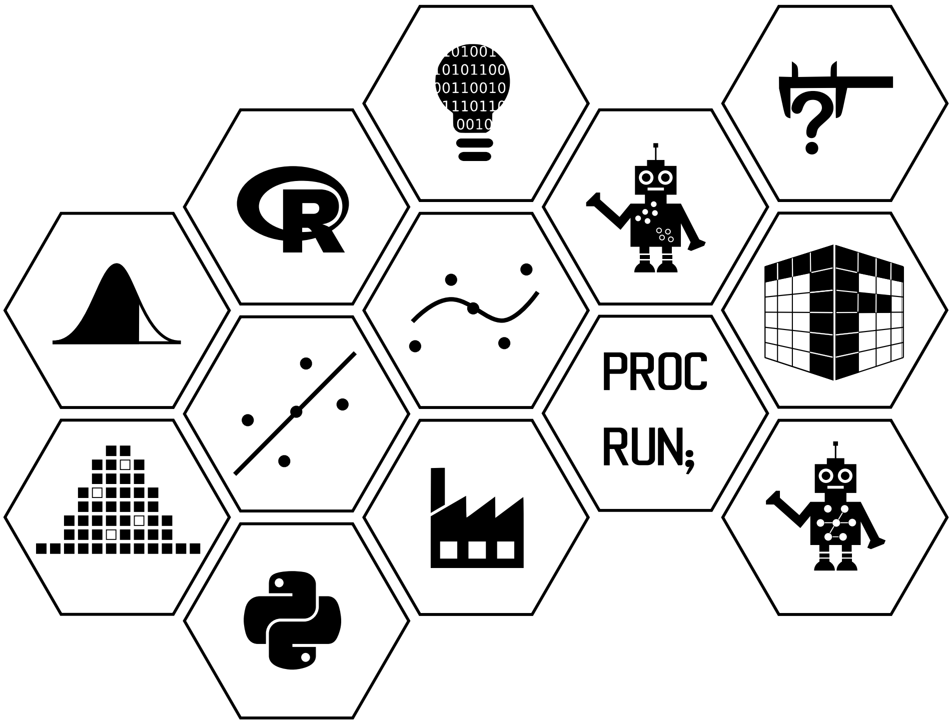
Unit 1:

Mathematical notation, sets, functions, exponents and logarithms



Mathematical notation and symbols

Introduction

This introductory block is here to remind you of some important notations and conventions used in Mathematics and Statistics.

Numbers and common notations

- The numbers $1, 2, 3, \ldots$ are called **natural numbers**. These are denoted by $\mathbb{N}$ (whereas the set $\mathbb{N}_0$ denotes all natural numbers including the number 0).
- **Integers** are denoted by $\mathbb{Z}$ and include negative numbers too: $\ldots, -2, -1, 0, 1, 2, \ldots$
- Numbers that can be expressed as a ratio of two integers (that is, of the form $\frac{a}{b}$ where a and b are integers, and $b \neq 0$) are said to be **rational**.
- Numbers such as $\sqrt{2}, \pi, e$ cannot be expressed as a ratio of integers; thus they are called **irrational**.
- The set of **real numbers** includes both rational and irrational numbers and is denoted by $\mathbb{R}$.
- The **reciprocal** of any number is found if we divide 1 by that number. For example, the reciprocal of 3 is $\frac{1}{3}$ and the reciprocal of $\frac{1}{3}$ is 3. Note that the old denominator has become the new numerator, and the old numerator has become the new denominator.
- The **absolute value** of a number can be thought of as its distance from zero. This is denoted by vertical lines around the number. For example, $|6|$ (read "the absolute value of 6") is 6, and $|-6|$ is 6 again.
- The **factorial** of a non-negative integer number n is denoted by $n!$ (read "n factorial") and is the product of all positive integers less than or equal to n. For example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. We also define $0!$ to be equal to 1.

Using symbols Mathematics provides a very rich language for the communication of different concepts and ideas. In order to use this language it is of high importance to appreciate how symbols are used to represent physical quantities, and to understand the rules and conventions that have been developed to manipulate them.

The choice of which letters or symbols to use is up to the user, although it is helpful to choose letters that have some meaning in any particular context. For example, if we wish to choose a symbol to represent the temperature in a room we might choose the capital letter T. Usually the lowercase letter t is used to represent time. Since both time and temperature can vary we refer to t and T as **variables**. In a particular calculation some symbols represent fixed and unchanging quantities and we call these **constants**.

We often reserve the letters x, y and z to stand for variables and use the earlier letters of the alphabet, such as a, b and c, to represent constants. The Greek letter π is used to represent the constant $3.14159\ldots$ which appears in the formula for the area of the circle. Other Greek letters are frequently used, and for reference the Greek alphabet is given below.

Letter	Upper case	Lower case	Letter	Upper case	Lower case
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ϵ or ε	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ or ϑ	Upsilon	Y	υ
Iota	I	ι	Phi	Φ	ϕ or φ
Kappa	K	κ	Chi	X	ξ
Lambda	Λ	λ	Psi	Ψ	χ
Mu	M	μ	Omega	Ω	ω

Mathematics is a very precise language and care must be taken to note the exact position of any symbol in relation to any other. If x and y are two symbols, then the quantities xy, x^y and x_y can all mean different things. In the expression x^y, y is called a superscript while in the expression x_y it is called a subscript.

- If the letters x and y represent two numbers, then their **sum** is written as $x + y$.

- Subtracting y from x yields $x - y$. This quantity is also called the **difference** of x and y.
- The instruction to multiply x and y is written as $x \times y$ where usually the multiplication sign is omitted and we simply write xy. This quantity is called the **product** of x and y.
- Note that xy is the same as yx. Because of this we say that multiplication is **commutative**.
- Multiplication is also **associative**. When we multiply three quantities together, such as $x \times y \times z$, it doesn't matter whether we evaluate $x \times y$ first and then multiply the result by z, or evaluate $y \times z$ first and then multiply the result by x. In other words, $(x \times y) \times z = x \times (y \times z)$.
- The quantity $\frac{x}{y}$ (or x/y) means that x is divided by y. In the expression $\frac{x}{y}$ the top line is called the **numerator** and the bottom line is called the **denominator**. Division by 1 leaves any number unchanged (i.e. $\frac{x}{1}$ is simply x) while division by 0 is never allowed.
- The equals sign, =, is used in several different ways:
  - It can be used in **equations**. The left-hand side and right hand side of an equation are equal only when the variable involved takes specific values known as solutions of the equation. For example, in the equation $x - 10 = 0$, the variable is x and the left-hand side and right-hand side are equal when x has the value 10. If x has any other value the two sides are not equal.
  - It can be used in **formulae**. Physical quantities are often related through a formula. For example, the formula of the length, C, of the circumference of a circle expresses the relationship between the circumference of the circle and its radius r. It specifically states that $C = 2\pi r$. When used in this way the equals sign expresses the fact that the quantity on the left is found by evaluating the expression on the right.
  - It can also be used in **identities**. At first sight an identity looks like an equation, except that is true for all values of the variable. For example, $(x-1)(x+1) = x^2 - 1$ is true for all values of the variable x.
- The sign $\neq$ is read "is not equal to". For example it is correct to write $12 \neq 21$.
- The $\sum$ notation (read **"Sigma notation"**) provides a convenient way of writing long sums. The sum $x_1 + x_2 + x_3 + \ldots + x_{20}$ is written using the capital Greek letter sigma, $\sum$, as $\sum_{i=1}^{i=20} x_i$.
- The $\prod$ notation (read **"product notation"**) provides a convenient way of writing long products. The product $x_1 \times x_2 \times x_3 \times \ldots \times x_{20}$ is written using the capital Greek letter Pi, $\prod$, as $\prod_{i=1}^{i=20} x_i$.

Inequalities Given any two real numbers a and b, there are three mutually exclusive possibilities:

- $a > b$ (a is greater than b),
- $a < b$ (a is less than b), or
- $a = b$ (a is equal to b).

The inequality in the first two cases is said to be **strict**.

The case where "a is greater than or equal to b" is denoted by $a \geq b$. Similarly, we have that $a \leq b$.

In these cases, the inequalities are said to be **weak**.

Some useful relations are:

- If $a > b$ and $b > c$; then $a > c$.
- If $a > b$; then $a + c > b + c$ for any c.
- If $a > b$; then $ac > bc$ for any positive c.
- If $a > b$; then $ac < bc$ for any negative c.

Laws of indices **Indices** or **powers** provide a convenient notation when we need to multiply a number by itself several times. the number $5 \times 5 \times 5$ is written as 5^3 and read "5 raised to the power of 3". Similarly we could have

$$8 \times 8 \times 8 \times 8 = 8^4, \quad (-2) \times (-2) = (-2)^2, \quad z \times z \times z \times z \times z = z^5.$$

More generally, in the expression x^y, x is called the **base** and y is called the **index** or **power**.

There are a number of rules that enable us to manipulate expressions involving indices. These rules are known as the laws of indices and they occur so commonly that it is worthwhile to memorise them.

The **laws of indices** state:

- $a^m \times a^n = a^{m+n}$ (when multiplying two numbers that have the same base we just add their indices)
- $\frac{a^m}{a^n} = a^{m-n}$ (when dividing two numbers that have the same base we subtract their indices)
- $(a^m)^n = a^{mn}$ (if a number is raised to a power and the result itself is raised to a power, the two powers are multiplied together)

Note that in all the previous rules the base was the same throughout.

Two important results that can be derived from these laws are that:

- $a^0 = 1$ (any number raised to the power of 0 is 1), and
- $a^1 = a$ (any number raised to the power of 1 is itself).

A generalisation of the third law states:

- $(a^m b^k)^n = a^{mn} b^{nk}$ (when two numbers, a^m and b^k, are multiplied together and they are raised to the same power, each number is raised to that power and they can then be multiplied together).

Negative indices A number can be raised to a negative power. This is interpreted as raising the reciprocal number to the positive power. For example, $5^{-2} = \left(\frac{1}{5}\right)^2 = \frac{1^2}{5^2} = \frac{1}{25}$.

Generally, we have that $a^{-m} = \frac{1}{a^m}$ and $a^m = \frac{1}{a^{-m}}$.

Fractional indices Let's now consider the expression $(16^{1/2})^2$. Using the third law of indices we can write it as

$$\begin{aligned}(16^{1/2})^2 &= 16^{\frac{1}{2}2}\\ &= 16^1\\ &= 16.\end{aligned}$$

So $16^{1/2}$ is a number that when it is raised to the power of 2 equals 16. That means that it could be 4 or -4. In other words $16^{1/2}$ is a square root of 16, that is $\sqrt{16}$. There are always two square roots of a non-zero number, and we write $16^{1/2} = \pm 4$.

Similarly, we have that

$$\begin{aligned}(8^{1/3})^3 &= 8^{\frac{1}{3}3}\\ &= 8^1\\ &= 8,\end{aligned}$$

so that $8^{1/3}$ is a number that when it is raised to the power of 3 equals 8. Thus $8^{1/3}$ is the cubic root of 8, that is $\sqrt[3]{8}$ which is equal to 2. Each number has only one cubic root.

Generally, we have that $x^{\frac{1}{n}}$ is the n-th root of x, that is defined as $\sqrt[n]{x}$. The generalisation of the third law of indices states that $(a^m b^k)^n = a^{mn} b^{nk}$. By taking $m = k = \frac{1}{2}$ and $n = 1$ we have that $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Polynomial expressions An important group of mathematical expressions that use indices are known as **polynomial expressions**. Examples of polynomials are

$$5x^3 - 3x^2 + 10, \quad 11 - 5x^5 + 7x, \quad y - y^3.$$

Notice that they are all constructed using non-negative whole-number powers of the variable. Recall that $x^0 = 1$ and so the number 10 appearing in the first expression can be thought of $10x^0$.

A polynomial expression takes the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where a_0, a_1, a_2, a_3 are all constants called the coefficients of the polynomial. The number a_0 is also called the constant term. The highest power in a polynomial is called the **degree** of the polynomial. Polynomials with degree $3, 2, 1$ and 0 are known as cubic, quadratic, linear and constant respectively.

Tasks *Answers to the tasks can be found at the end of the booklet, but please attempt the tasks first.*

Task 1.

Write out explicitly what is meant by the following:

(a) $\sum_{i=1}^{i=6} k^i$

(b) $\sum_{i=1}^{i=6} i^k$

(c) $\sum_{i=1}^{i=6} (i+1)^k$

(d) $\sum_{i=1}^{i=6} 2$

(e) $\prod_{i=1}^{i=6} k^i$

(f) $\prod_{i=1}^{i=6} 2$

Task 2.

By writing out the terms explicitly show that

$$\sum_{i=1}^{i=5} 3i = 3\sum_{i=1}^{i=5} i.$$

Task 3.

Write out fully, the following expressions:

(a) $3m^4$

(b) $(3m)^4$

Task 4.

Simplify the following expressions:

(a) $b^5 \times b^2 \times b$

(b) $b^5 \times b^2 \times \frac{b}{b^3}$

Task 5.

Remove the parentheses from the following expressions:

(a) $(3x)^2$

(b) $(6xy)^4$

(c) $(x^3y^5)^3$

Task 6.

Show that $(-xy)^3$ is equal to $-x^3y^3$.

Task 7.

Write each of the following expressions using a positive index:

(a) 2^{-3}
(b) $\frac{1}{4^{-3}}$
(c) x^{-5}

Task 8.

Simplify the following expressions:

(a) $\frac{a^8a^3}{a^5}$
(b) $\frac{a^8b^2a^3b^4}{b^7a^5}$

Task 9.

Evaluate the following:

(a) $144^{1/2}$
(b) $125^{1/3}$

Task 10.

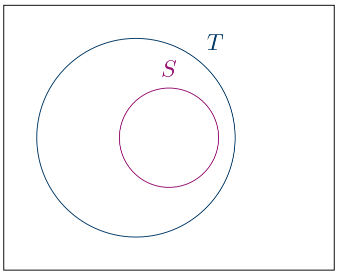
Simplify the following:

(a) $\frac{\sqrt{x}}{x^3x^2}$
(b) $\frac{x^2}{x^{-1/2}\sqrt[3]{x^2}}$

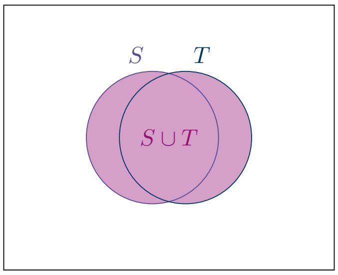
Sets

Introduction to sets

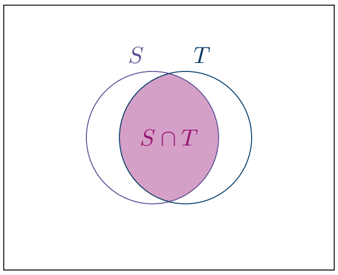
- A **set** S is a well-defined, unordered collection of objects. We typically use curly brackets to denote sets, for example $S = \{1, 2\}$.
- The objects that make up the set are also known as **elements** of the set.
- If x is an element of S, we can say that x belongs to S and write $x \in S$ (the symbol $\in$ reads "belongs to" or "in"). If, on the other hand, an element z does not belong to S we can write $z \notin S$. To give an example, for $S = \{1, 2\}$, $1 \in S$, but $3 \notin S$.
- A set may contain finitely many or infinitely many elements.
- A set with no elements is called the **empty set** and is denoted by the symbol $\emptyset$.
- The number of elements within a set S is called the **cardinality** of the set and is denoted by $\mathbf{card}(S)$ or $|S|$.
- Given sets S and T , we say that S is a **subset** of T if every element of S is also an element of T. We can then write $S \subset T$. In that case, we can also say that T is a **superset** of S; and write it as $T \supset S$. The diagram below (which is known as a Venn diagram) illustrates the definition.



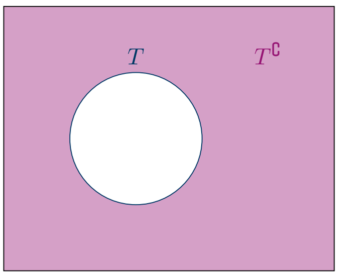
- Given sets S and T, their **union** $S \cup T$ is the set of elements that are either in S or T (or in both).



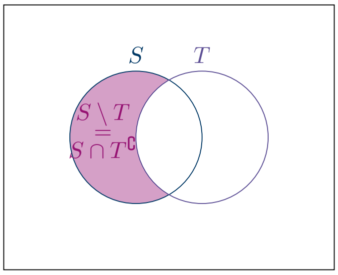
- Given sets S and T, their **intersection** $S \cap T$ is the set of elements that are both in S and T.



- A set S is called the **complement** of T if it contains all the elements that do not belong to it. The complement of T is written as $T^{\complement}$ (or $\overline{T}$ or T').



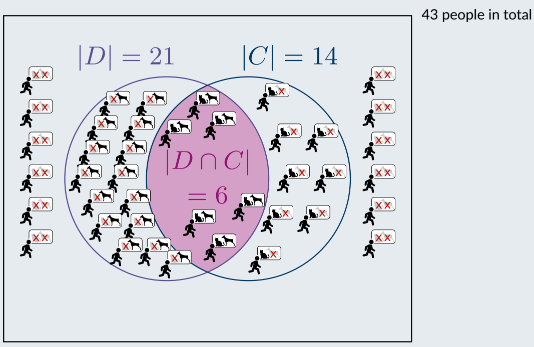
- Given two sets, S and T, the **difference** $S \setminus T$ contains all elements of S that are not contained in T. The set difference can be, more formally defined as the intersection of S and the complement of T, $S \setminus T = S \cap T^{\complement}$.



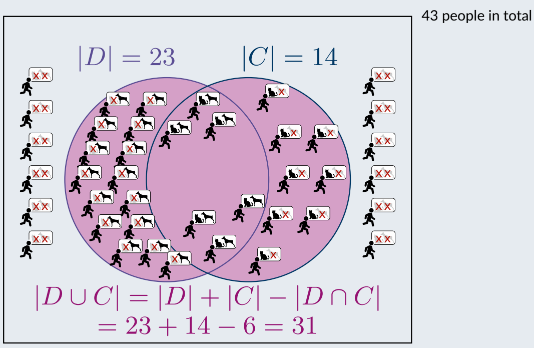
Example 1 (Dog people vs Cat people).

Let's assume that I asked 43 people if they like dogs or cats. 23 of them said they like dogs, 14 of them told me they like cats while there were 6 people who like both dogs and cats.

If we denote as C and D the sets referring to the people who like cats and dogs respectively; then we are given the following information: $|D| = 23$, $|C| = 14$ and $|D \cap C| = 6$. Also, there are are 43 people in total. This information is shown on the diagram below (which is known as a Venn diagram).

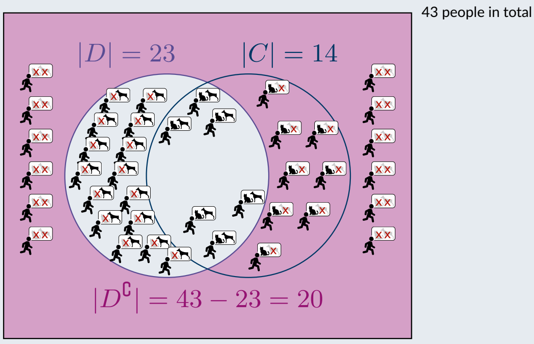


We will first use the Venn diagram to work out the number of people who like dogs or cats (or both), i.e. we will find the number of elements (cardinality) of the union $D \cup C$.



31 people like dogs, cats or both. Note that when we calculated the number people who like dogs or cats, we had to subtract the number of people who like both dogs and cats. The reason for this is that when we add the number of people who like dogs to the number of people who like cats, we have counted those who like both dogs and cats twice. Hence we have to subtract their number, so that we count everyone only once.

Next we will use the Venn diagram to find the number of people who do not like dogs, i.e. the cardinality of the complement $D^{\complement}$.



20 people do not like dogs.

Example 2 (European capitals).

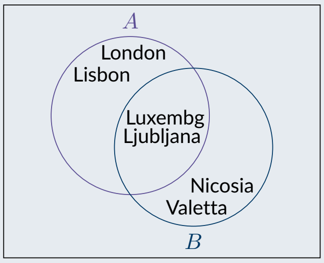
Let's assume that we want to look at the set A of some European capital cities that start with the letter L. In this case we have

$$A = \{\text{Lisbon, Ljubljana, London, Luxembourg}\}.$$

Now, let's assume we are interested to look at the set of the European capital cities that have a small population (let's say less than 300000 people). In this case we have

$$B = \{\text{Ljubljana, Valetta, Nicosia, Luxembourg}\}.$$

The corresponding Venn diagram is shown below.



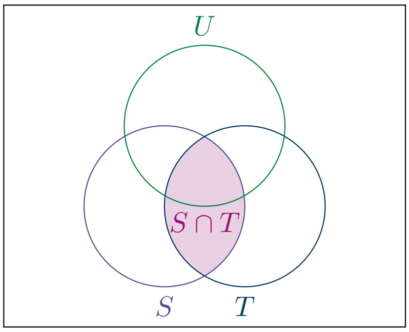
Let's now look at some rule for manipulating expressions involving sets.

- For any two sets, S and T, the intersection of S and T is the same as the intersection of T and S, $S \cap T = T \cap S$. Similarly, the union of S and T is the same as the union of T and S, $S \cup T = T \cup S$. This property (which also holds for addition and multiplication of real numbers) is called **commutativity**. Note that the set difference is not commutative: $S \setminus T$ is *not* the same as $T \setminus S$ (you can illustrate this on a Venn diagram).
- For any three sets, S, T and U,

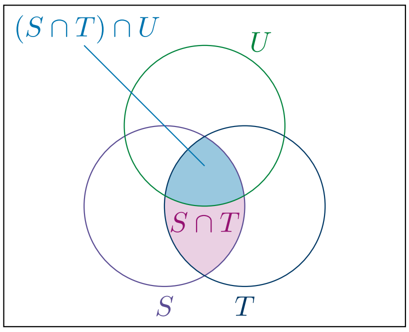
$$(S \cap T) \cap U = S \cap (T \cap U),$$

i.e. the order in which we take intersections does not matter. This property is called **associativity** in Mathematics. We can use Venn diagrams to illustrate why this property holds. The left column below identifies $(S \cap T) \cap U$, whereas the right column identifies $S \cap (T \cap U)$. We can see both are the same.

To calculate $(S \cap T) \cap U$ we first calculate $S \cap T$ …



…and then calculate the intersection of $S \cap T$ and U.



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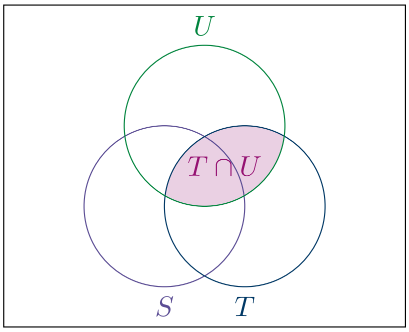
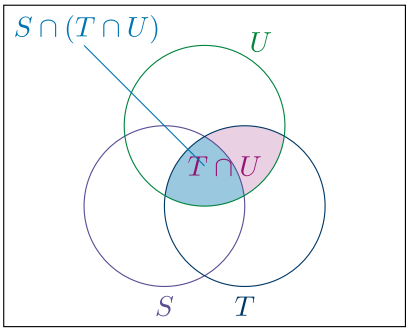
To calculate $S \cap (T \cap U)$ we first calculate $T \cap U$ …



…and then calculate the intersection of S and $T \cap U$.

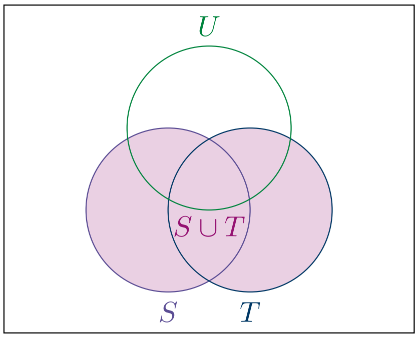


- Similarly, for any three sets, S, T and U,

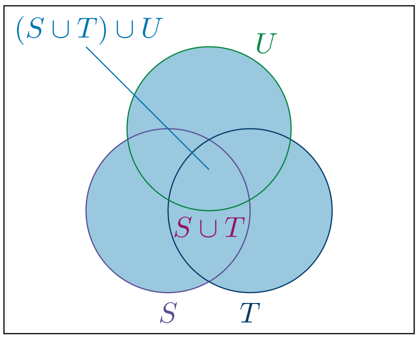
$$(S \cup T) \cup U = S \cup (T \cup U),$$

i.e. the order in which we take unions does not matter either. This property is called **associativity** in Mathematics.

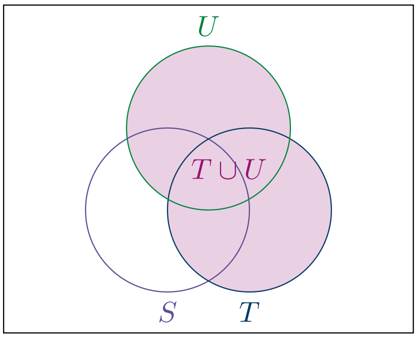
To calculate $(S \cup T) \cup U$ we first calculate $S \cup T$ …



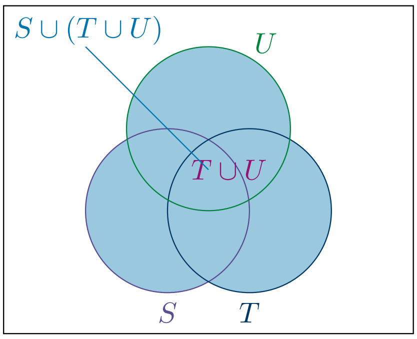
…and then calculate the union of $S \cup T$ and U.



To calculate $S \cup (T \cup U)$ we first calculate $T \cup U$ …



…and then calculate the union of S and $T \cup U$.

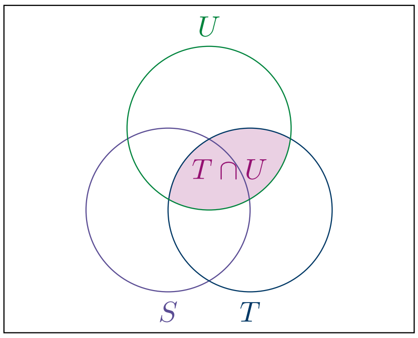


- Furthermore, for any three sets, S, T and U,

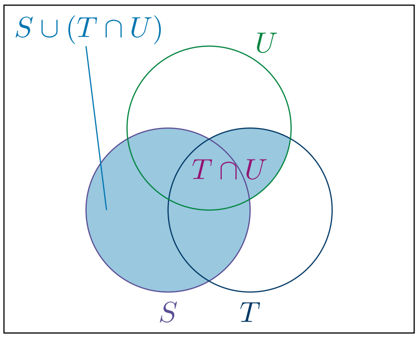
$$S \cup (T \cap U) = (S \cup T) \cap (S \cup U).$$

This property is called **distributivity** in Mathematics. We can again try to understand this rule by identifying both the left-hand sided and the right-hand side in a Venn diagram.

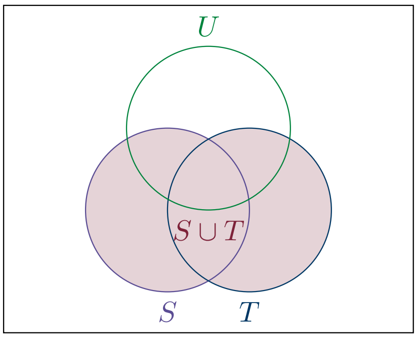
To calculate $S \cup (T \cap U)$ we first calculate $T \cap U$ …



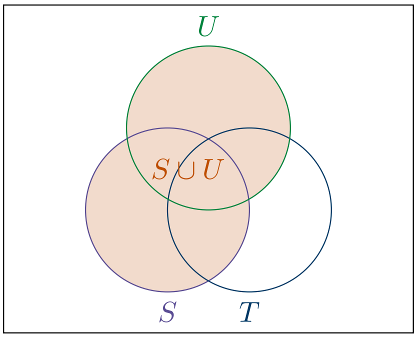
…and then calculate the union of S and $T \cap U$.



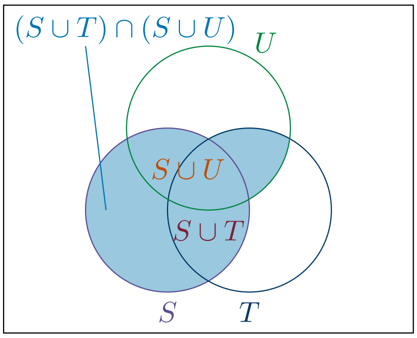
To calculate $(S \cup T) \cap (S \cup U)$ we first calculate $S \cup T$ …



…as well as $S \cup U$ …



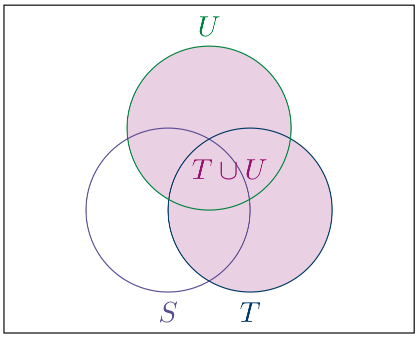
…and then calculate the interection of these two sets, $S \cup T$ and $S \cup U$.



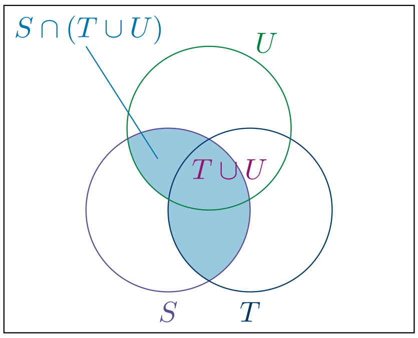
- Similarly, for any three sets, S, T and U,

$$S \cap (T \cup U) = (S \cap T) \cup (S \cap U).$$

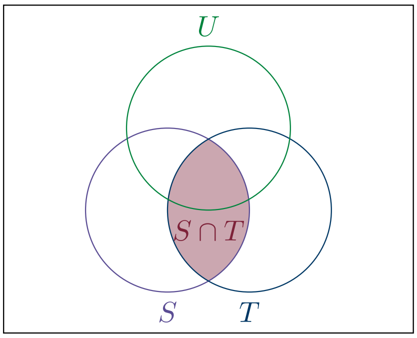
To calculate $S \cap (T \cup U)$ we first calculate $T \cup U$ …



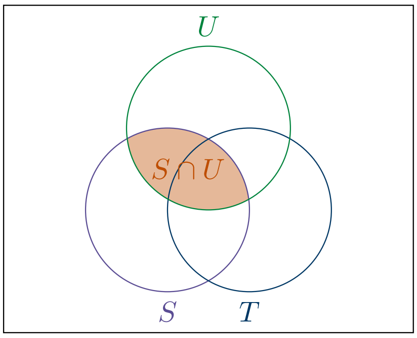
…and then calculate the intersection of S and $T \cup U$.



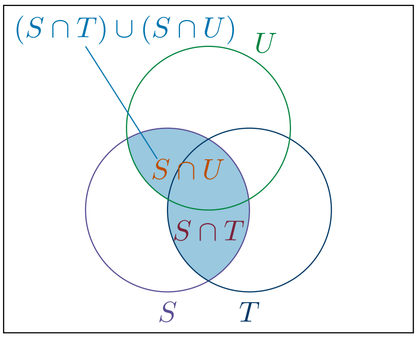
To calculate $(S \cap T) \cup (S \cap U)$ we first calculate $S \cap T$ …



…as well as $S \cap U$ …



…and then calculate the union of these two sets, $S \cap T$ and $S \cap U$.



You might at first be slightly puzzled by these rules, but you have already been familiar with most of them. You know them from doing arithmetic with numbers. Just think of unions as additions and intersections as multiplications.

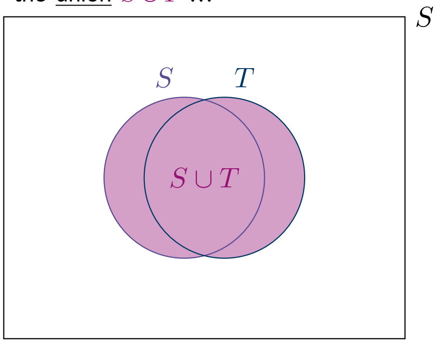
Name	Rule for sets	Corresponding rule for add'n and mult'n of numbers	Example
Commutativty	$S \cap T = T \cap S$	$s \times t = t \times s$	$3 \times 4 = 12 = 4 \times 3$
	$S \cup T = T \cup S$	$s + t = t + s$	$3 + 4 = 7 = 4 + 3$
Associativity	$(S \cap T) \cap U = S \cap (T \cap U)$	$(s \times t) \times u = s \times (t \times u)$	$(2 \times 3) \times 4 = 6 \times 4 = 24 = 2 \times 12 = 2 \times (3 \times 4)$
	$(S \cup T) \cup U = S \cup (T \cup U)$	$(s + t) + u = s + (t + u)$	$(2 + 3) + 4 = 5 + 4 = 9 = 2 + 7 = 2 + (3 + 4)$
Distributivity	$S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$	*No equivalent rule*	*Addition is not distributive over multiplication.*
	$S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$	$s \times (t + u) = s \times t + s \times u$	$2 \times (3 + 4) = 2 \times 7 = 14 = 6 + 8 = 2 \times 3 + 2 \times 4$

The final two rules are important rules, but have no equivalent in terms of addition or multiplication. They are called De Morgan's laws.

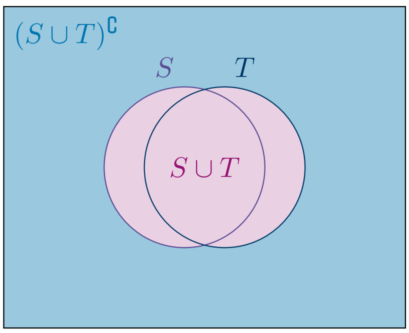
- The complement of the union of two sets equals the intersection of their complements, i.e.

$$(S \cup T)^{\mathsf{C}} = S^{\mathsf{C}} \cap T^{\mathsf{C}}.$$

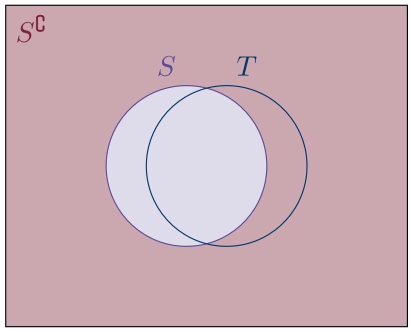
To calculate $(S \cup T)^{\mathsf{C}}$ we first calculate the <u>union</u> $S \cup T$ ...



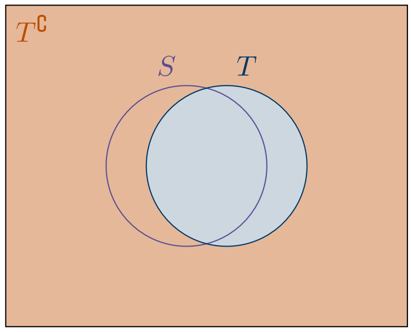
...and then take the complement.



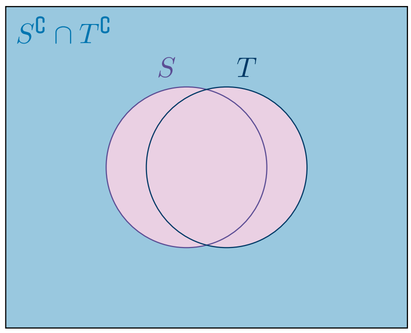
To calculate $S^{\mathsf{C}} \cap T^{\mathsf{C}}$ we first take the complement of S ...



...as well as the complement of T ...



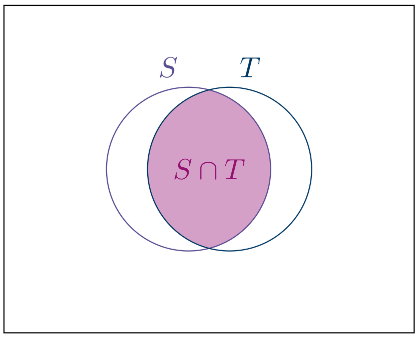
...and then take the <u>intersection</u> of the two complements S^{C} and T^{C}.



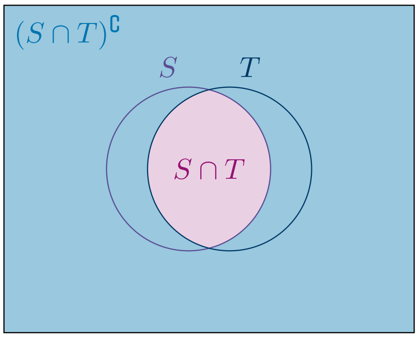
- De Morgan's laws also state that the complement of the intersection of two sets equals the union of their complements, i.e.

$$(S \cap T)^{\complement} = S^{\complement} \cup T^{\complement}.$$

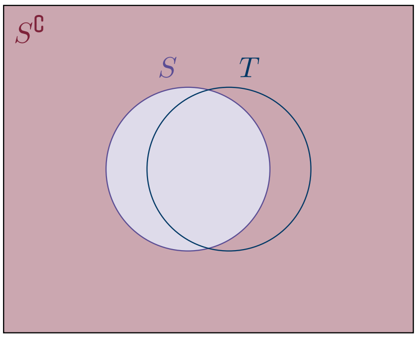
To calculate $(S \cap T)^{\complement}$ we first calculate the intersection $S \cap T$ …



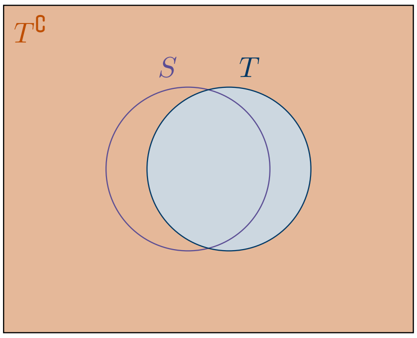
…and then take the complement.



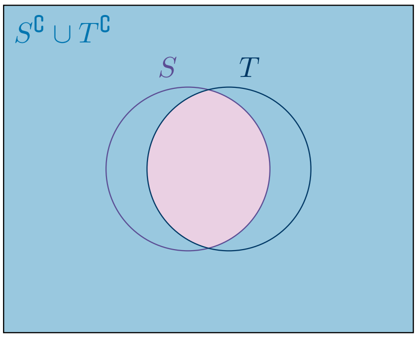
To calculate $S^{\complement} \cup T^{\complement}$ we first take the complement of S …



…as well as the complement of T …



…and then take the union of the two complements $S^{\complement}$ and $T^{\complement}$.



Again, you have already been familiar with these rules. This time not from arithmetic with numbers, but from logical statements involving the English words "not" (complement), "or" (union) as well as "and" (intersection).

De Morgan Law	Logical equivalent	Example
$(S \cup T)^{\complement} =$ $S^{\complement} \cap T^{\complement}$	`not (S or T) =` `(not S) and (not T)`	"Alice does not like shellfish *or* tuna." is equivalent to "Alice does not like shellfish *and* she does not like tuna."
$(S \cap T)^{\complement} =$ $S^{\complement} \cap T^{\complement}$	`not (S and T) =` `(not S) or (not T)`	"Bob is not both short *and* tall." is equivalent to "Bob is either not short *or* he is not tall."

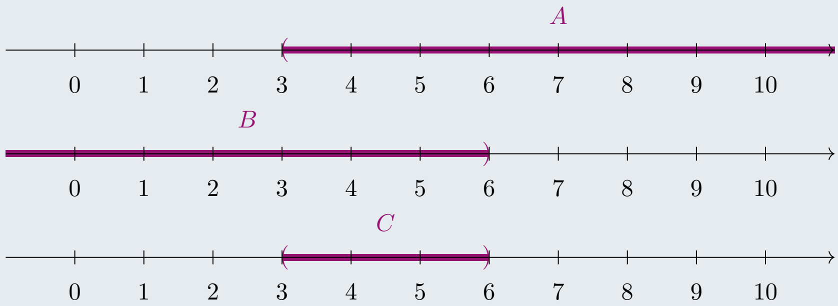
Actually, all the rules we have seen so far, not just De Morgan's laws (including commutativity, associativity and distributivity) hold for logical statements involving "and" and "or".

So far we have defined all sets by listing their entries. We can also define sets by stating a property that lets us determine what is an element of the set. For example, the set S consisting of all number which are at least 1 can be written as $\{x : x > 1\}$.

Example 3 (Intervals).

If we want the set to be comprised of all the numbers that are greater than 3, we have $A = \{x : x > 3\}$. Similarly, if we want a set which consists of all the numbers that are smaller than 6, we have $B = \{x : x < 6\}$. Finally, $C = \{x : 3 < x < 6\}$ consists of all the numbers between 3 and 6.

As the sets A, B, and C are intervals it is easiest to illustrate them on the real line.



All intervals are infinite sets: they contain an infinite number of elements. The reason for this is the infinite resolution of the real numbers. Between any two real numbers lie an infinite number of other real numbers.

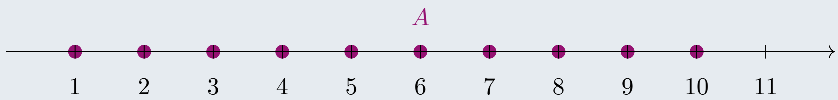
We will come back to intervals at the end of this section.

Example 4 (Sets containing integers).

Assume that set $A = \{\text{Positive integers less than } 11\}$.
We now look at two sets U and V, which are subsets of A: U consists of all multiples of 3 whereas V consists of all multiples of 2.

Let's first try to write down the elements of A. They are

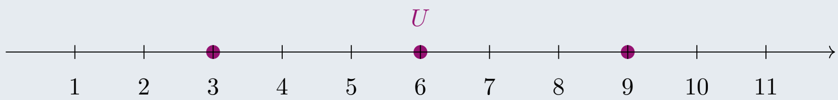
$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$



Note that A is a finite set of integers, and not an interval.

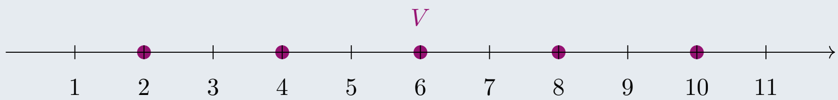
The elements of U are the elements of A which are multiples of 3:

$$U = \{3, 6, 9\}$$



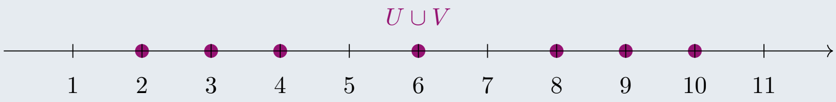
The elements of V are the elements of A which are multiples of 2:

$$V = \{2, 4, 6, 8, 10\}$$



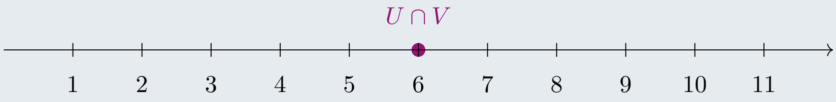
The elements of the union $U \cup V$ are those numbers which are multiples of 2 or 3:

$$U \cup V = \{3, 6, 9\} \cup \{2, 4, 6, 8, 10\} = \{2, 3, 4, 6, 8, 9, 10\}$$



The elements of the intersection $U \cap V$ are those numbers which are multiples of 2 and of 3:

$$U \cap V = \{3, 6, 9\} \cap \{2, 4, 6, 8, 10\} = \{6\}$$



Bounded sets A set S of real numbers is **bounded above** if there exists a real number H that is greater than or equal to every element of the set. That is, for some H we have $x \leq H$ for all $x \in S$. The number H, if it exists, is called the **upper bound** of the set S.

A set of real numbers is **bounded below** if there exists a real number h that is less than or equal to every element of the set. That is, $x \geq h$ for all $x \in S$. The number h, if it exists, is called the **lower bound** of the set S.

A set that is bounded below and bounded above is called a **bounded set**.

Example 5 (Bounded sets).

(a) The set of all real numbers $\mathbb{R}$ is neither bounded below, nor bounded above. For every real number x we can think of, there is another real number (for example $x+1$), which is larger than it. Similarly, for every real number x we can think of there is another real number (for example $x - 1$), which is smaller than it.
(b) The set of all natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ is bounded below, as all natural numbers n are greater or equal to 1. Just like the real numbers, $\mathbb{N}$ is not bounded above.
(c) Just like the real numbers, the set of all integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, is neither bounded below, nor bounded above.
(d) The set $\{\frac{1}{n} | n \in \mathbb{N}\}$, however, is both bounded above and bounded below. 0 is a lower bound for the set, and 1 is a upper bound for the set, as

$$0 < \frac{1}{n} \leq 1, \qquad \text{for all } n \in \mathbb{N}$$

(e) The set $\{x : x > 2\}$ is bounded below, but not bounded above.

Maximum and minimum If a set S:

- has a largest element M, we call M the **maximum** element of the set.
- has a smallest element m, we call m the **minimum** element of the set.

Example 6 (Maximum and minimum).

(a) The set of all real numbers $\mathbb{R}$, neither has a minimum, nor does it have a maximum.
(b) The set of all natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ has the minimum 1, as no natural numbers are less than 1.
(c) The set of all integers $\mathbb{N} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ has neither minimum or maximum.
(d) The set $\{\frac{1}{n} : n \in \mathbb{N}_{>0}\}$ is a very interesting case. We have just seen that it has a lower bound (0) and an upper bound (1). Whilst 1 is also its maximum, it has no minimum, despite having a lower bound. As n increases, $\frac{1}{n}$ is getting smaller and smaller, but it will never be exactly 0.
(e) The set $\{x : x > 2\}$ has no minimum either, despite having a lower bound (2). The lower bound 2 is not an element of the set. Note that this would be different if we looked at the set $\{x : x \geq 2\}$. In that case, the lower bound 2 is an element of the set (as we have replaced the $>$ sign by a $\geq$ sign), and hence 2 is the minimum of the set.

Can you see a relationship between the existence of bounds and of a minimum or maximum?

The general rules are:

- Finite sets have a maximum if and only if they have an upper bound. Similarly, they have a minimum if and only if they have a lower bound.

- Infinite sets are more complicated. If an infinite set has a minimum (maximum) then it also has a lower bound (upper bound), but the converse is not true. We have seen infinite sets that were bounded below but that did not have a minimum. Similarly, there can be infinite sets that are bounded above but do not have a maximum.

Intervals If a and b are two real numbers, the set of all numbers that lie in between a and b is called an **interval**. An **open interval** does not contain its **boundary points**. The following is an open interval

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

A **closed interval** contains its **boundary points**. The following is a closed interval

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

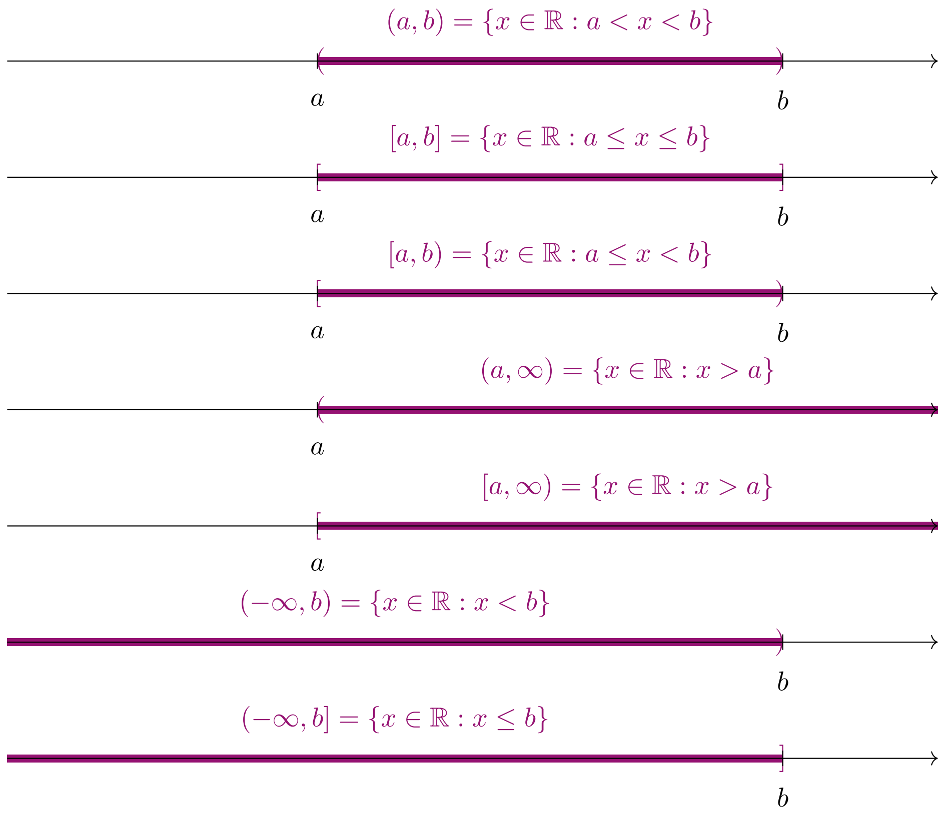
Finally, we may have a **half-open interval** like

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

The intervals listed previously are all bounded. The following intervals are all unbounded.

$$(a, \infty) = \{x \in \mathbb{R} : a < x\},$$
$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\},$$
$$(-\infty, b) = \{x \in \mathbb{R} : x < b\},$$
$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}.$$

The figure below illustrates the definition of these intervals.



Tasks

Task 11.

Let the set $A = \{1, 2, 5, 6\}$ and $B = \{2, 5, 6, 8\}$. Which of the following is true?

(a) A is a subset of B
(b) B is a subset of A
(c) A is a subset of B and B is a subset of A
(d) Neither A is a subset of B nor B is a subset of A

Task 12.

For $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 9\}$, find $A \cup B$.

Self help

Sets and types of numbers

http://www.math-exercises.com/sets-and-types-of-numbers

More exercises on sets

http://www.onlinemathlearning.com/math-sets.html

Worksheet on sets

http://www.math-only-math.com/worksheet-on-operation-on-sets.html

Venn diagrams: an introduction (YouTube video)

https://www.youtube.com/watch?v=YAjxRUGS0Gc

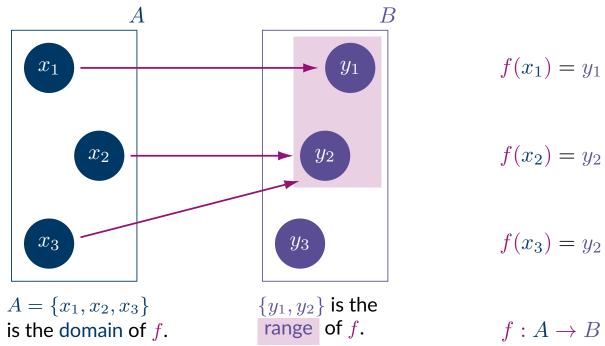
Sets: Union and intersection (YouTube video)

https://www.youtube.com/watch?v=En8fI2ixepo

Functions

Introduction to functions

- A **function** is a rule that associates a unique value with any element of a set. A function, $f(x)$, from a set A to a set B defines a rule that assigns for each $x \in A$ a unique element $f(x) \in B$.
- It can also be thought of as a rule that operates on an **input** x, sometimes called the **argument** of the function, and produces an **output** $f(x)$.
- In order for a rule to be a function it must produce a single output for any given input. It is however possible that two (or more) inputs are mapped to the same output.
- The set of all values that it "maps" from is called the **domain**.
- The set of values it maps to is called the **range**. The mapping can be denoted as $f(x) : A \to B$ where A and B are the domain and the range of the function $f(x)$ respectively.



Example 7.

Suppose we want to have an output that is 3 times the input; we can define the function $f(x) = 3x$. In that case, we might also say that the input x can only take positive values. That refers to the domain of the function. To find the range of the function, we have to find what are all the possible outputs. Since we are always multiplying a positive number with the number 3 that means that we will always end up with a positive number. That is the range of the function. To check if this a function or not we can clearly see that for any specific value of x we will always get the same output (e.g. if $x = 3$ then $f(3)$ is always equal to 9). The value of the output is often called the value of the function.

Example 8 (Converting temperature from Celsius to Fahrenheit degrees).

Let's consider the rule that maps temperature measurements from the Celsius scale to the Fahrenheit scale

$$f(x) = 1.8x + 32$$

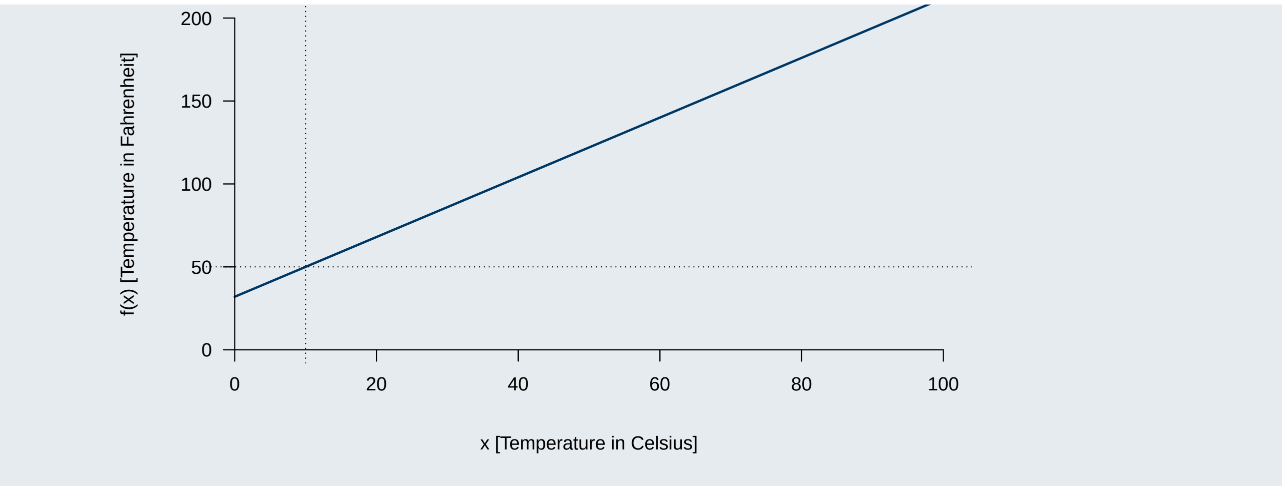
where x is the Celsius measurement and $f(x)$ the associated value in Fahrenheit.

So, if the temperature in Glasgow is 10 Celsius degrees and we want to find the equivalent temperature in Fahrenheit; we just need to find $f(10)$. Replacing x with the value 10 in the previous function will give us $f(10) = 50$ Fahrenheit degrees.

In literature, it is common to use y instead of $f(x)$. It is just a different notation, nothing else changes.

Graphs are a convenient and widely used way of portraying functions. By inspecting a graph it is easy to describe a number of properties of the function being considered. For example, where is the function positive? and where is is negative? Is it increasing or decreasing?

In order to plot the graph you can start using different values for the argument x and then write down the values of the function $f(x)$. Afterwards, you can draw a pair of axes and connect all the previous $(x, f(x))$ pairs.

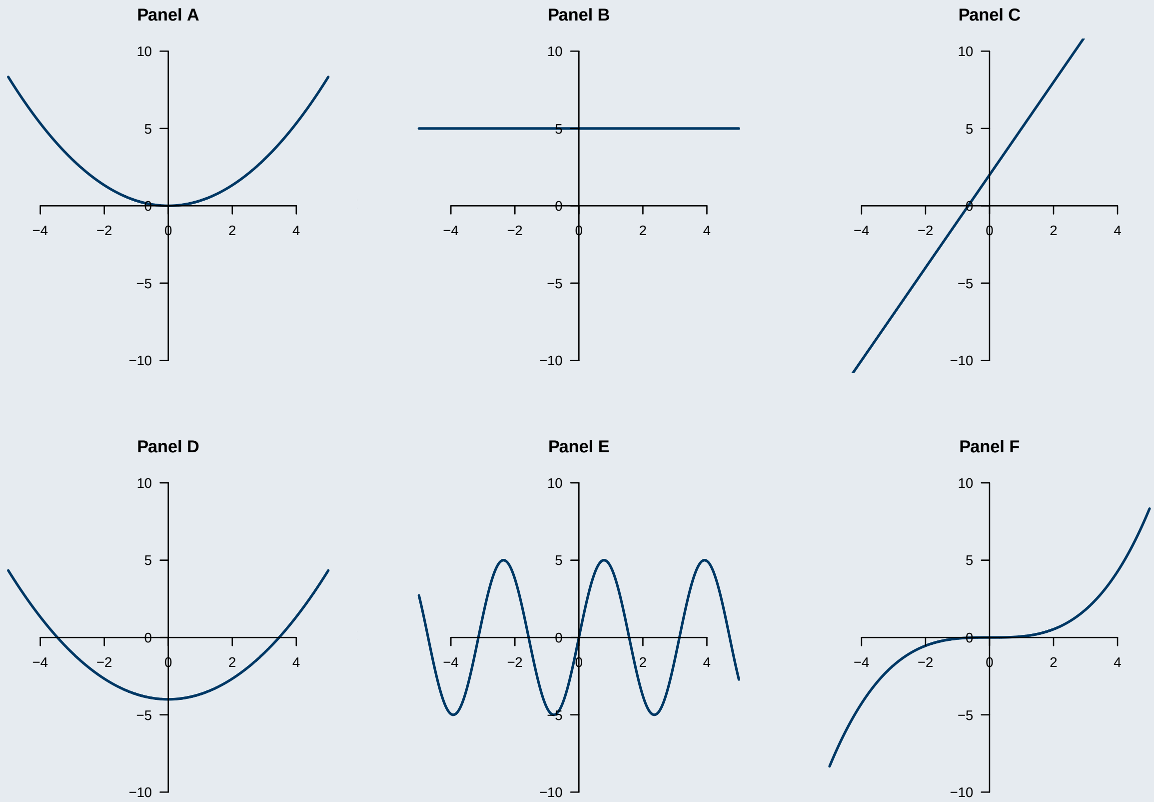


Standard classes of functions

- **Algebraic functions**: are functions that can be expressed as the solution of a polynomial equation with integer coefficients. Some examples of **polynomial functions** are:
  - **Constant function** $f(x) = a$,
  - **Identity function** $f(x) = x$,
  - **Linear function** $f(x) = ax + b$,
  - **Quadratic function** $f(x) = a + bx + cx^2$,
  - **Cubic function** $f(x) = a + bx + cx^2 + dx^3$.
- **Transcendental functions** are functions that are not algebraic. Some examples are:
  - **Exponential function** $f(x) = e^x$,
  - **Logarithmic function** $f(x) = \log(x)$,
  - **Trigonometric functions** like $f(x) = -3\sin(2x)$.

Example 9 (Connect functions to their graphs).

Below are the graphs of six functions.



We will now try to match them to the following six functions:

$$f(x) = 2 + 3x \qquad f(x) = \frac{x^2}{3} \qquad f(x) = \frac{x^2}{3} - 4$$

$$f(x) = 5\sin(2x) \qquad f(x) = \frac{x^3}{15} \qquad f(x) = 5$$

- The function at the top-left (Panel A) is a parabola. Two functions of the functions listed above are also parabolae, so we need to find out which one is which. We can see from the plot that f(0)=0, hence this must be $f(x) = \frac{x^2}{3}$.
- The function to its right (Panel B) is constant, and the only constant function is $f(x) = 5$.
- The function at the top-right (Panel C) is a linear function with a positive slope (i.e. the function increases as x is increasing), i.e. it must be $f(x) = 2 + 3x$.
- The function at the bottom-left (Panel D) is another parabola, this time with $f(0) < 0$. Hence, it must be $f(x) = \frac{x^2}{3} - 4$.
- The function to its right (Panel E) is an oscillating function, so it must be the trigonometric function $f(x) = 5\sin(2x)$.
- The function at the bottom-right (Panel F) is a non-linear odd function, i.e. it satisfies $f(-x) = -f(x)$. Hence it must be $f(x) = \frac{x^3}{15}$.

Inverse of a function We have seen that a function can be regarded as taking an input x, and processing it in some way to produce a single output $f(x)$. A natural question is whether we can find a function that will reverse the process. If we can find such a function it is called an inverse function to $f(x)$ and is given the symbol $f^{-1}(x)$. Do not confuse the -1 with an index or a power. Here, the superscript is used purely as the notation for the inverse function.

To find the inverse of a function you can:

1. Replace $f(x)$ with y (this will make the rest of the process easier).
2. Replace every x with a y and replace every y with an x.
3. Solve the equation from Step 2 for y, and finally
4. replace y with $f^{-1}(x)$.

Can you find the inverse of the function

$$f(x) = 1.8x + 32$$

where x is the Celsius measurement and $f(x)$ the associated value in Fahrenheit?

(*Hint*: You will have to end up with a function where the input x is the Fahrenheit measurement and $f(x)$ is the associated value in Celsius.)

A function $f(x)$, defined on domain A, is **one-to-one** if $f(x)$ never has the same value for two distinct points in A. This means that if we choose two different values x_1 and x_2 such that $x_1 \neq x_2$; then we will have $f(x_1) \neq f(x_2)$. Functions such as $f(x) = x^2$ do not possess an inverse since there are two values of x associated with each $f(x)$ (e.g. if we use $x_1 = 1$ and $x_2 = -1$ we have that $f(x_1) = f(x_2)$). Note that every one-to-one function has an inverse.

Tasks

Task 13.

Consider the function defined as $f(x) = x^2 - \frac{6}{x}, x \neq 0$. Find $f(3)$.

Task 14.

If $f(x) = \sqrt{x-6}$ (for $x > 6$), show that $f^{-1}(x) = x^2 + 6$.

Task 15.

If $f(x) = \frac{x+10}{3x}$ (for $x \neq 0$), then for what value x is $f(x) = y$? (i.e. solve $f(x) = y$ for x)

Task 16.

Find the inverse of the following functions,

(a) $f(x) = 8x + 1$
(b) $f(x) = \frac{2x-7}{x}$ where $x \neq 0$
(c) $h(x) = \sqrt[3]{6x-12} + 1$

Self help

Introduction to functions

http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-introfns-2009-1.pdf

What is a function (Khan Academy video)

https://www.khanacademy.org/math/algebra/algebra-functions/evaluating-functions/v/what-is-a-function

Exercises on functions from OpenStax Precalculus

https://math.libretexts.org/Bookshelves/Precalculus/Book%3A_Precalculus_(OpenStax)/01%3A_Functions/1.0E%3A_1.E%3A_Functions_(Exercises)

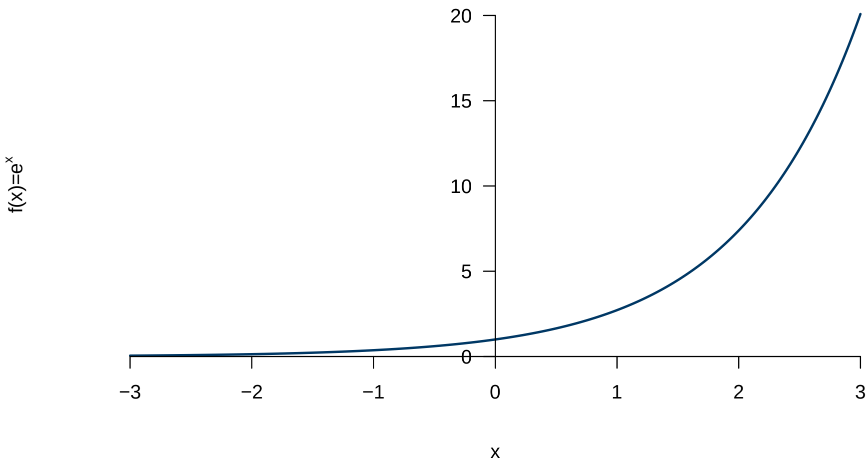
Exponents and logarithms

Introduction to exponents

An exponent is another name for a power (or an index). Expressions involving exponents are called **exponential expressions**. For example, consider any positive real number a. We define the **exponential function** to base a as $f(x) = a^x$.

Sometimes a specific irrational number $e \approx 2.7183$ is used as the base for the exponential function (called as the **natural exponential function**). In that case we have $f(x) = \exp(x) = e^x$. The domain of the function is all the real numbers $\mathbb{R}$ (this can be also written as $(-\infty, \infty)$ while the range of the function is only the positive real numbers $x > 0$.

The figure below shows the graph of the natural exponential function for $x \in [-3, 3]$.



Since the exponential function is monotonic it has an inverse. Its inverse function is the natural logarithm, but more on that later.

Some properties of the exponential function can be seen from the figure:

1. As x becomes large and positive, e^x increases without bound. We express this mathematically as $e^x \to \infty$ as $x \to \infty$ (the symbol $\to$ reads "goes to" and the symbol ∞ refers to infinity).
2. As x becomes large and negative, e^x approaches 0. We write $e^x \to 0$ as $x \to -\infty$.
3. The function e^x is never negative.

The property that e^x increases as x increases is referred to as **exponential growth**.

Laws of exponents

The laws of indices and the rules of algebra apply to exponential expressions.

- $e^0 = 1$
- $e^m e^n = e^{m+n}$
- $\frac{e^m}{e^n} = e^{m-n}$
- $e^{-m} = \frac{1}{e^m}$
- $(e^m)^n = e^{mn}$

Introduction to logarithms

Logarithms are an alternative way of writing expressions that involve powers, or indices.

Consider the expression $36 = 6^2$. Remember that 6 is the base and 2 is the power. Another way to write this expression is $\log_6(36) = 2$ and is stated as "log to base 6 of 36 is 2". We see that the logarithm, 2, is the same as the power in the original expression. The base in the original expression is the same as the base of the logarithm.

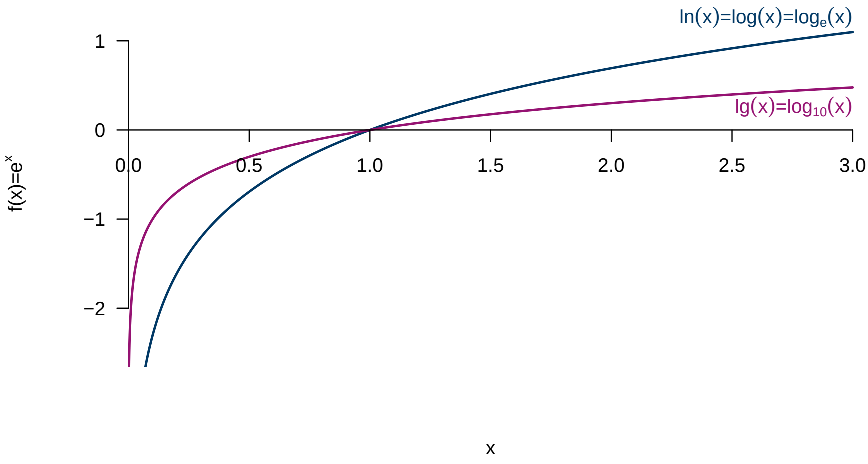
The two statements

$$36 = 6^2 \quad \log_6(36) = 2$$

are equivalent. If we write one of them, we are automatically implying the other.

In general, if a is a positive constant and $N = a^x$ then $\log_a(N) = x$. The number a is called the **base** of the logarithm. In practice most logarithms are to the base 10 or e. The latter logarithms, to base e, are called natural logarithms and are usually denoted by $\ln$ or $\log$ (without a subscript, though some use $\log$ to refer to the logarithm with base 10, which we will denote by $\lg(x)$).

We define the **logarithmic function** to base a as $f(x) = \log_a(x)$. If we use e as the base we then have the **natural logarithmic function** that we write as $f(x) = \log_e(x) \equiv \log(x) \equiv \ln(x)$. We can see the results of plotting the graphs of the logarithmic (with base 10) and the natural logarithmic function for $x \in [0, 3]$ in the figure below.



Some properties of the logarithmic function can be seen from the figure:

1. As x increases, both $\lg(x)$ and $\ln(x)$ increase indefinitely. We write this mathematically as $\lg(x) \to \infty$ as $x \to \infty$, $\ln(x) \to \infty$ as $x \to \infty$.
2. As x approaches 0 both $\lg(x)$ and $\ln(x)$ approach minus infinity. We express this as $\lg(x) \to -\infty$ as $x \to 0$, $\ln(x) \to -\infty$ as $x \to 0$.
3. The value for both functions is 0 when the argument x takes the value 1.
4. Both functions are not defined when x is negative or zero. The domain of the function is all the positive real numbers, $x > 0$, while the range of the function is all real numbers $\mathbb{R}$.

Laws of logarithms Just as expressions involving indices can be simplified using appropriate laws, so expressions involving logarithms can be simplified using the laws of logarithms. These laws hold true for any base. However it is essential that the same base is used throughout an expression before the laws can be applied.

- $\log_a(1) = 0$
- $\log_a(mn) = \log_a(m) + \log_a(n)$
- $\log_a(\frac{m}{n}) = \log_a(m) - \log_a(n)$
- $\log_a(\frac{1}{m}) = -\log_a(m)$
- $\log_a(m^n) = n\log_a(m)$

Tasks

Task 17.

Write the following using logarithms:

(a) $32 = 2^5$
(b) $4^3 = 64$
(c) $10^2 = 100$
(d) $0.001 = 10^{-3}$
(e) $e^{-1.3} = 0.2725$

Task 18.

Write the following using indices:

(a) $\log_5(625) = 4$
(b) $\log_2(256) = 8$
(c) $\ln(17) = 2.83$

Task 19.

Evaluate $\ln(\frac{1}{\sqrt{e}})$

Task 20.

Evaluate $\log_{10}(\frac{1}{1000})$

Task 21.

If $\ln(x)$ - $\ln(4)$ = 0, then what value does x take?

Task 22.

Simplify $\frac{1}{2}(2\ln(3) - 2\ln(7))$.

Self help

Exercises on logarithms and exponents

https://www.shmoop.com/logarithm-exponent/exponent-log-properties-exercises.html

Quadratic Equations

Quadratic Functions We will take a closer look at quadratics, which are polynomials of degree 2. You may come across quadratics in different forms, such as:

- an expression $ax^2 + bx + c$
- a function $f(x) = ax^2 + bx + c$
- an equation $ax^2 + bx + c = 0$
- etc.

where a, b and c are constants and $a \neq 0$

Solving Quadratic Equations These can be solved by factorising, using the quadratic formula or by completing the square. We will focus on the first two methods only.

Solving quadratic equations by factorising:

To solve a quadratic equation $ax^2 + bx + c = 0$, let's first start by factorising $ax^2 + bx + c$. To do this we need to find two numbers, α and β, that multiply to give us c (i.e., factors of c) and add to give us b so that we end up with:

$$ax^2 + bx + c = (x + \alpha)(x + \beta)$$

It's easier to illustrate this using an example.

Example 10.

Suppose we wish to solve the equation

$$x^2 + 4x + 3 = 0$$

We first start by factorising $x^2 + 4x + 3$. Two numbers that multiply to give 3 and add to give 4, are 1 and 3.

So we have:

$$x^2 + 4x + 3 = (x + 3)(x + 1)$$

The original equation can be rewritten in terms of the product of the factors:

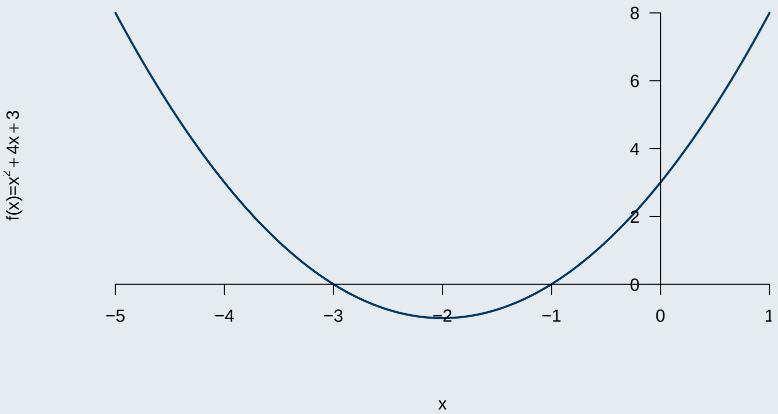
$$\begin{aligned} x^2 + 4x + 3 &= 0 \\ (x + 3)(x + 1) &= 0 \end{aligned}$$

We now have two factors, $(x + 3)$ and $(x + 1)$ multiplying together to give us 0. If we multiply two (or more) factors and get a zero result, then we know that at least one of the factors is itself equal to zero. So we have:

$x + 3 = 0$ or $x + 1 = 0$
so $x = -3$ or $x = -1$

$x = -1$ and $x = -3$ are called the **roots** of the quadratic equation $x^2 + 4x + 3 = 0$.

Visually, finding the roots of a quadratic corresponds to finding the intersections of the graph of the quadratic function $f(x) = x^2 + 4x + 3$ with the x-axis.



Let's try a few more examples.

Example 11.

Solve $x^2 + 2x - 8 = 0$.

In this example we note that the second sign is negative, which means that the signs in the brackets must be different (unlike the previous example where the signs in both brackets were positive).

The pairs of numbers multiplying to give 8 are $1, 8$ or $2, 4$.

Since the numbers add to give $+2$, we go with $+4$ and -2.

So

$$x^2 + 2x - 8 = 0$$

factorises to give

$$(x + 4)(x - 2) = 0$$

This means that:

$$\begin{aligned} & x + 4 = 0 \quad \text{or} \quad x - 2 = 0 \\ \text{so} \quad & x = -4 \quad \text{or} \quad x = 2 \end{aligned}$$

Example 12.

Solve $x^2 - 16x + 15 = 0$.

We note that the numbers in the brackets should multiply to give $+15$, which means both numbers should be positive or both should be negative.

The pairs of numbers multiplying to give 15 are $1, 15$ or $3, 5$.

Since the numbers add to give -16, we choose -1 and -15.

So

$$x^2 - 16x + 15 = 0$$

factorises to give

$$(x - 1)(x - 15) = 0$$

This means that:

$$\begin{aligned} & x - 1 = 0 \quad \text{or} \quad x - 15 = 0 \\ \text{so} \quad & x = 1 \quad \text{or} \quad x = 15 \end{aligned}$$

Example 13.

Solve $x^2 - 2x - 24 = 0$

Here we note that the numbers in the brackets should multiply to give -24. The pairs of numbers multiplying to give 24 are $1, 24$; $2, 12$; $3, 8$ or $4, 6$.

Since the numbers add to give -2, we choose -6 and $+4$. So

$$x^2 - 2x - 24 = 0$$

factorises to give

$$(x - 6)(x + 4) = 0$$

This means that:

$$x - 6 = 0 \quad \text{or} \quad x + 4 = 0$$

$$\text{so} \quad x = 6 \quad \text{or} \quad x = -4$$

Example 14.

Solve $x^2 - 8x + 16 = 0$ The pairs of numbers multiplying to give 16 are $1, 16; 2, 8$ and $4, 4$. We go with $-4, -4$ since they multiply to give $+16$ and add to give -8.

So

$$x^2 - 8x + 16 = 0$$

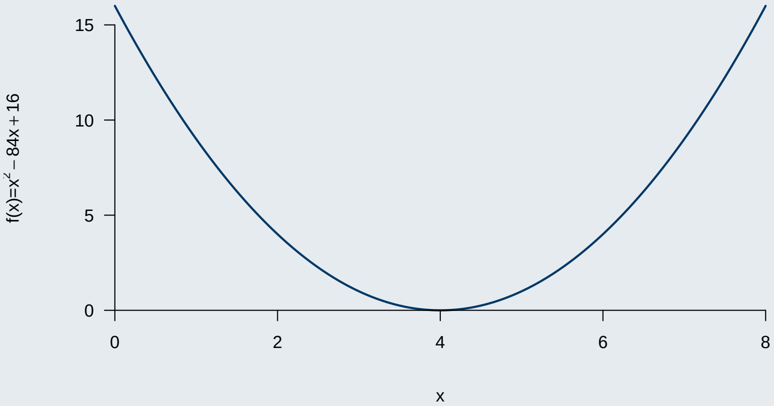
factorises to give

$$(x - 4)(x - 4) = 0$$

This means that:

$$x - 4 = 0 \quad \text{and} \quad x = 4$$

Here we have an example of **repeated roots**. Visually, this corresponds to the graph of the quadratic function $f(x) = x^2 - 8x + 16$ touching (but not intersecting) the x-axis at the repeated root.



Using the Quadratic Formula Many quadratic equations cannot be solved by factorisation easily, sometimes because they do not have simple factors. The way round this is to use the **quadratic formula**. The solution of an equation $ax^2 + bx + c = 0$ is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The symbol ± means that the square root has a positive and a negative value, both of which must be used in solving for x.

Let's try out an example that cannot be easily factorised.

Example 15.

Solve $5x^2 - 11x - 4 = 0$.

We substitute $a = 5$, $b = -11$ and $c = -4$ into the quadratic formula and get:

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(5)(-4)}}{2(5)}$$

$$= \frac{11 \pm \sqrt{201}}{10}$$

Here we have $x = 2.52$ or -0.32 to 2d.p.

Task 23.

Have a go at some of these questions, using the quadratic formula if needed and rounding to 2d.p. if appropriate:

(a) $x^2 + 12x - 13 = 0$
(b) $x^2 + 2x + 1 = 0$
(c) $x^2 - x - 42 = 0$
(d) $3x^2 - 8x + 5 = 0$
(e) $x^2 - 6x + 3 = 0$

For a quadratic equation $ax^2 + bx + c = 0$, $(b^2 - 4ac)$ is called the **discriminant**.

- If $(b^2 - 4ac) > 0$, the roots are real and distinct;
- If $(b^2 - 4ac) = 0$, the roots are real and repeated;
- If $(b^2 - 4ac) < 0$, the roots are complex;

This last option takes us into the realm of Complex Numbers. For example, consider the quadratic

$$x^2 + 1 = 0$$

This means that $x^2 = -1$.

At this point, one would say that as we cannot find the square root of a negative number, this problem cannot be solved. However, a way to overcome this issue is that we define an imaginary number i as

$$i = \sqrt{-1}$$

and this allows us to now find x. Bearing in mind that $i = \sqrt{-1}$ (or that $i^2 = -1$), we can say that

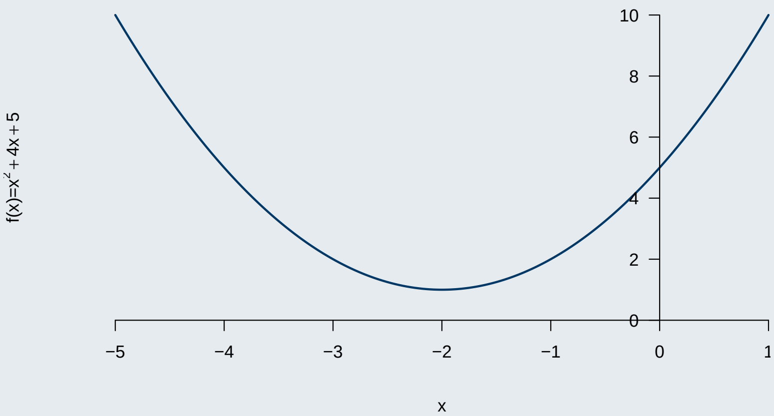
$$x = \pm i$$

Numbers formed by combining the imaginary number i with real numbers are called **complex numbers**. These numbers take the form $x + iy$ where x is the real part and y is the imaginary part; x and y are real numbers and $i = \sqrt{-1}$ is the imaginary number.

Example 16.

Consider the quadratic $x^2 + 4x + 5 = 0$.

If we draw the function we can see that its minimum lies above the x-axis, i.e. it never intersects the x-axis.



Using the quadratic formula, we get:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm 2i}{2} \\ &= -2 \pm 2i \end{aligned}$$

Notice that the roots have the same real part and different signs on the imaginary part. We call these **conjugate pairs**.

Self help

Khan Academy: Solving quadratics by factorising

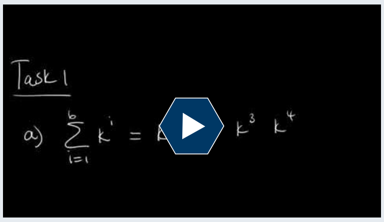
https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:quadratic-functions-equations/x2f8bb11595b61c86:quadratics-solve-factoring/v/example-1-solving-a-quadratic-equation-by-factoring

Khan Academy: Using the Quadratic formula

https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:quadratic-functions-equations/x2f8bb11595b61c86:quadratic-formula-a1/v/using-the-quadratic-formula

Answers to tasks

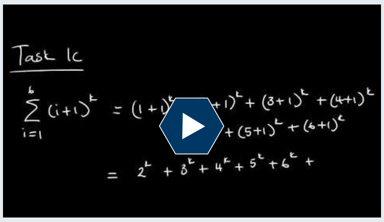
Answer to Task 1.



Video model answers for part (a)

https://youtu.be/Em3Z8Eyg4RM

Duration: 0m35s



Video model answers for part (c)

https://youtu.be/5BeJj3A0lEM

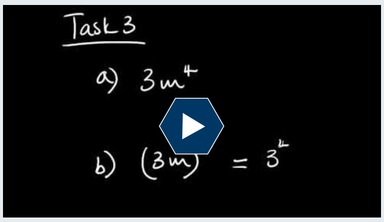
Duration: 1m01s

(a) $k^1 + k^2 + k^3 + k^4 + k^5 + k^6$
(b) $1^k + 2^k + 3^k + 4^k + 5^k + 6^k$
(c) $2^k + 3^k + 4^k + 5^k + 6^k + 7^k$
(d) $2 + 2 + 2 + 2 + 2 + 2 = 12$
(e) $k^1 \times k^2 \times k^3 \times k^4 \times k^5 \times k^6 = k^{1+2+3+4+5+6} = k^{21}$
(f) $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$

Answer to Task 2.

$$\sum_{i=1}^{i=5} 3i = 3 + 6 + 9 + 12 + 15 = 3 \times 1 + 3 \times 2 + 3 \times 3 + 3 \times 4 + 3 \times 5 = 3 \times (1 + 2 + 3 + 4 + 5) = 3 \sum_{i=1}^{i=5} i$$

Answer to Task 3.



Video model answers

https://youtu.be/50Kiw4mZq1A

Duration: 0m42s

(a) $3m^4$
(b) $81m^4$

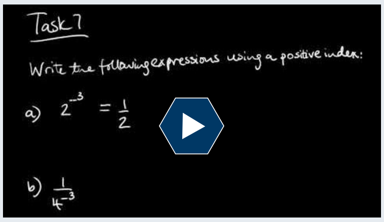
Answer to Task 4. (a) b^8
(b) b^5

Answer to Task 5. (a) $9x^2$
(b) $6^4x^4y^4$
(c) x^9y^{15}

Answer to Task 6.

$$(-xy)^3 = (-1)^3x^3y^3 = -x^3y^3.$$

Answer to Task 7.



Video model answers

https://youtu.be/lPB8BO-GQfY

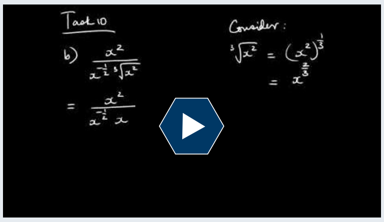
Duration: 0m42s

(a) $\frac{1}{2^3}$
(b) 4^3
(c) $\frac{1}{x^5}$

Answer to Task 8. (a) a^6
(b) $\frac{a^6}{b}$

Answer to Task 9. (a) 12
(b) 5

Answer to Task 10.



Video model answers

https://youtu.be/nUm_vb2Qf2s

Duration: 1m29s

(a) $\frac{1}{x^{\frac{9}{2}}} = \frac{1}{\sqrt{x^9}}$
(b) $x^{11/6}$

Answer to Task 11. The correct answer is (d). Neither A is a subset of B nor is B a subset of A.

Answer to Task 12. $A \cup B = \{1, 2, 3, 5\} \cup \{2, 4, 5, 9\} = \{1, 2, 3, 4, 5, 9\}$

Answer to Task 13. $f(3) = 3^2 - \frac{6}{3} = 9 - 2 = 7$.

Answer to Task 14. We could find the inverse by solving $f(x) = y$ for x, but it is easier to show that $f^{-1}(f(x)) = x$, i.e. applying the function $f^{-1}(x)$ "undoes" the effect of applying the function $f(x)$.

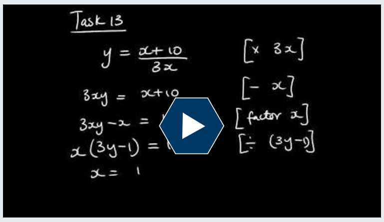
$$f^{-1}(f(x)) = f(x)^2 + 6 = (\sqrt{x-6})^2 + 6 = x - 6 + 6 = x$$

Similarly,

$$f(f^{-1}(x)) = \sqrt{f^{-1}(x) - 6} = \sqrt{x^2 + 6 - 6} = x$$

(Note that $x > 6$, i.e. we know that x cannot be negative.)

Answer to Task 15.



Video model answers for part

https://youtu.be/9tDuHq6rQ7o

Duration: 1m17s

If

$$y = f(x) = \frac{x + 10}{3x},$$

then this is (assuming that $x \neq 0$) equivalent to

$$y \times 3x = x + 10$$

We can now bring all terms involving x to the left-hand side and keep all terms not involving x on the right-hand side, giving

$$3xy - x = 10$$

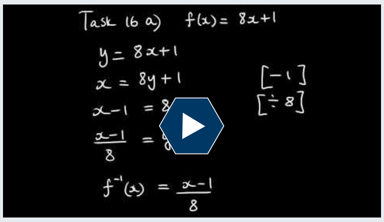
We can now pull out the common factor of x on the left-hand side, giving

$$x(3y - 1) = 10$$

Dividing both sides by $3y - 1$ gives

$$x = \frac{10}{3y - 1}$$

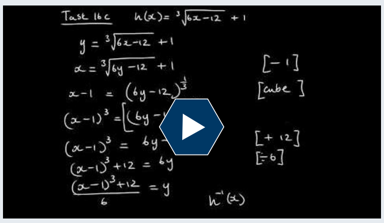
Answer to Task 16.



Video model answers for part (a)

https://youtu.be/gqCKV72Z-nk

Duration: 1m01s



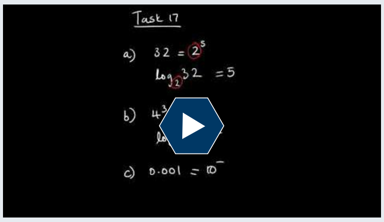
Video model answers for part (c)

https://youtu.be/q1rkScVtyUI

Duration: 2m11s

(a) $\frac{x-1}{8}$
(b) $\frac{7}{2-x}$
(c) $\frac{(x-1)^3+12}{6}$

Answer to Task 17.



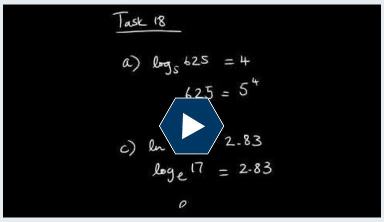
Video model answers for part

https://youtu.be/w2cR5ASjF48

Duration: 0m55s

(a) $\log_2 32 = 5$
(b) $\log_4 64 = 3$
(c) $\log_{10} 100 = 2$
(d) $\log_{10} 0.001 = -3$
(e) $\ln 0.2725 = -1.3$

Answer to Task 18.



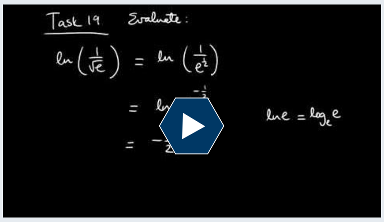
Video model answers for

https://youtu.be/NkzocKoCGpg

Duration: 0m50s

(a) $5^4 = 625$ (b) $2^8 = 256$ (c) $e^{2.83} = 17$

Answer to Task 19.



Video model answers for

https://youtu.be/ko7M3tDP3J4

Duration: 0m51s

$$\ln\left(\frac{1}{\sqrt{e}}\right) = \ln(e^{-\frac{1}{2}}) = -\frac{1}{2}\underbrace{\ln(e)}_{=1} = -\frac{1}{2}$$

Answer to Task 20.

$$\log_{10}\left(\frac{1}{1000}\right) = \log_{10}(10^{-3}) = -3\underbrace{\log_{10}(10)}_{=1} = -3$$

Answer to Task 21. Using that $\ln(x) - \ln(4) = \ln(\frac{x}{4})$, the statement is equivalent to

$$\ln(\frac{x}{4}) = 0$$

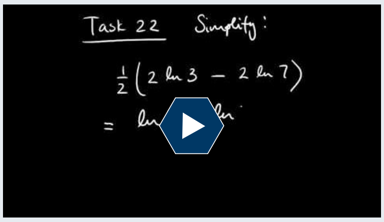
Exponentiating both sides gives

$$\frac{x}{4} = 1,$$

yielding

$$x = 4.$$

Answer to Task 22.



Video model answers for

https://youtu.be/WMO99W0Yito

Duration: 0m38s

$$\frac{1}{2}\big(2\ln(3) - 2\ln(7)\big) = \ln(3) - \ln(7) = \ln(\frac{3}{7})$$

Answer to Task 23. The answers to the first two parts are:

(a) Using the quadratic formula with $a = 1$, $b = 12$ and $c = -13$

$$
\begin{aligned}
rclx &= \frac{-12 \pm \sqrt{12^2 - 4(1)(-13)}}{2(1)} \\
&= \frac{-12 \pm \sqrt{196}}{2} \\
&= -6 \pm 7,
\end{aligned}
$$

so we have $x = -13$ and 1.

We could have also factorised $x^2 + 12x - 13 = (x-1)(x+13)$ (as $-1 + 13 = 12$ and $-1 \times 13 = -13$), immediately revealing the roots.

(b) Using the quadratic formula with $a = 1$, $b = 2$ and $c = 1$

$$
\begin{aligned}
x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{0}}{2} \\
&= -1 \pm 0
\end{aligned}
$$

so we have a repeated root at $x = -1$.

We could have also factorised $x^2 + 2x + 1 = (x+1)^2$ (as $1 + 1 = 2$ and $1 \times 1 = 1$), immediately revealing the repeated root.

The answers to the other parts are:

(c) $x = -6$ or $x = 7$.
(d) $x = \frac{5}{3} \approx 1.67$ or $x = 1$.
(e) $x = 3 - \sqrt{6} \approx 0.55$ or $x = 3 + \sqrt{6} \approx 5.45$.