Preliminary Mathematics for online MSc programmes in Data Analytics

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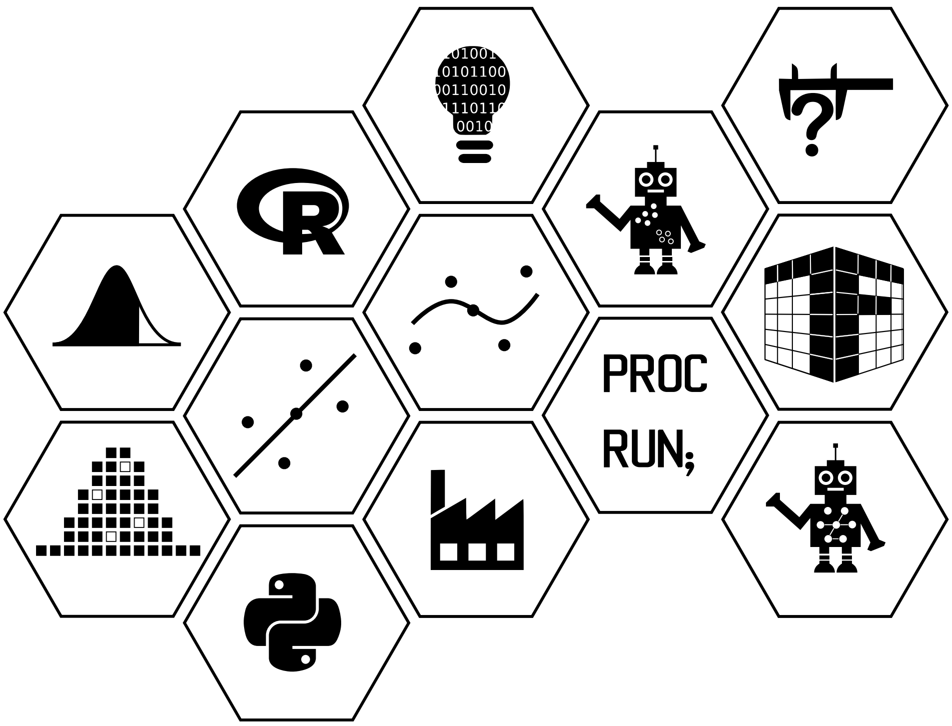
Unit 2:

Differentiation in 1D (minima and maxima)



DATA ANALYTICS GLASGOW

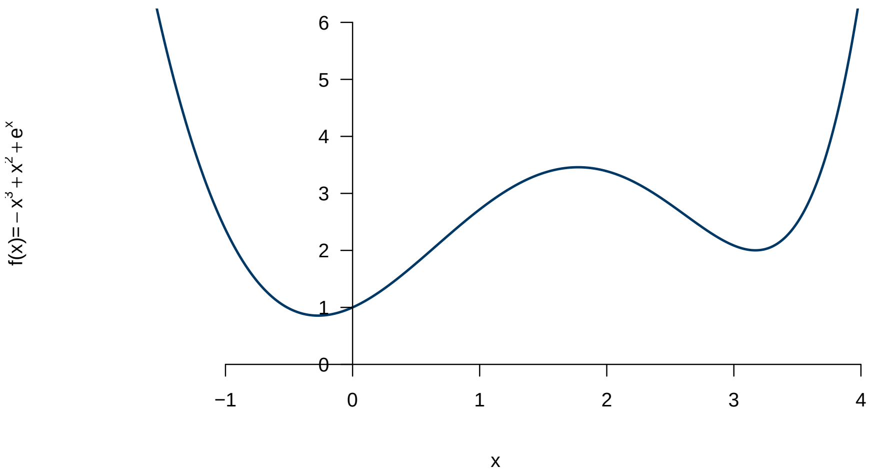
Differentiation

Introduction to differentiation

We are often interested in the rate at which some variable is changing. For example, we may be interested in the rate at which the temperature is changing in a chemical reaction or in the rate at which the pressure in a vessel is changing. Rapid rates of change of a variable may indicate that a system is not operating normally and is approaching critical values.

Rates of change may be positive, zero, or negative. A positive rate of change means that the variable is increasing; a zero rate of change means that the variable is not changing; while a negative change of rate means that the variable is decreasing.

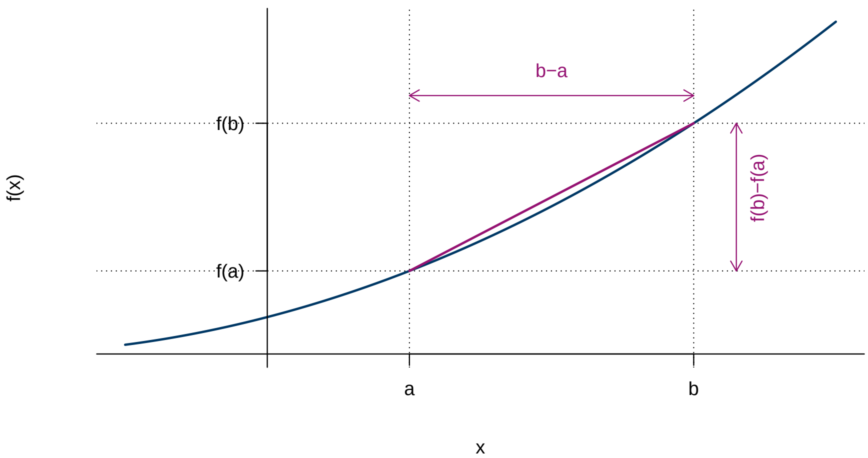
Consider the function $f(x) = -x^3 + x^2 + e^x$ for $x \in [-1.5, 4]$, shown below.



Between $x = -2$ and $x = -1$, the function is decreasing rapidly. Across this interval the rate of change of the function $f(x)$ is large and negative. Between $x = -1$ and $x = -0.3$ the function is still decreasing but not as rapidly as before. Across this interval the rate of change of the function $f(x)$ is small and negative. There is a small interval, $(-0.3, -0.2)$ that the function seems to not change at all. Across that interval the rate of change is zero. Between $x = -0.2$ and $x = 1.8$ the function is increasing rapidly; the rate of change is large and positive.

It is often not sufficient to describe a rate of change as "large and positive" or "small and negative". A precise value is needed. The technique for calculating the rate of change of any function is called **differentiation**. Use of differentiation provides a precise value or expression for the rate of change of a function.

Average rate of change across an interval We have already seen that a function can have different rates of change at different points on its graph. Let's first define and calculate the **average rate of change** of a function across an interval and later on we will also define the rate of change at a point. The figure below shows a function $f(x)$; two possible argument values, a and b, and their two respective outputs $f(a)$ and $f(b)$.



Consider that x is increasing from a to b. The change in x is $b - a$. As x increases from a to b, then the function $f(x)$ increases from $f(a)$ to $f(b)$. The change in $f(x)$ is $f(b) - f(a)$. Then the average rate of change of y across the interval is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

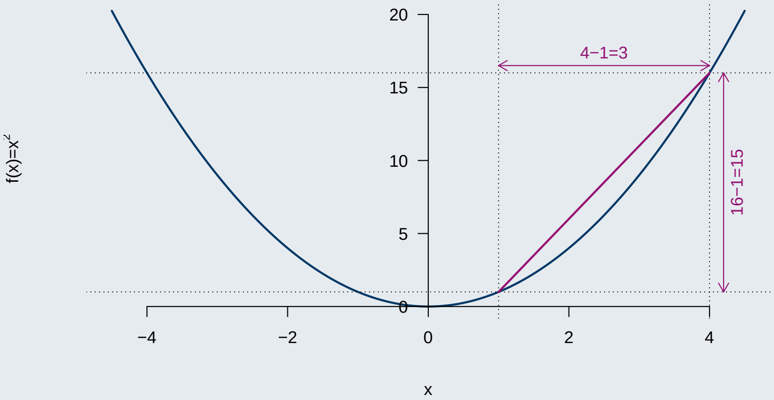
Another way to think of the average rate of change of a function is by visualising it as the slope of a line that passes through two points on the function. This line, called a **secant line**, can be drawn on a graph of a function so that we can quantify the value of the slope of the line. A secant line passing through the points $(a, f(a))$ and $(b, f(b))$ has a vertical rise of $f(b) - f(a)$ and a horizontal run of $b - a$. The slope of the line, between the points a and b, is $\frac{f(b)-f(a)}{b-a}$ (which is exactly the same as the average rate of change).

Example 1 (Average rate of change across an interval).

Let's calculate the average rate of change of $f(x) = x^2$ across the following intervals

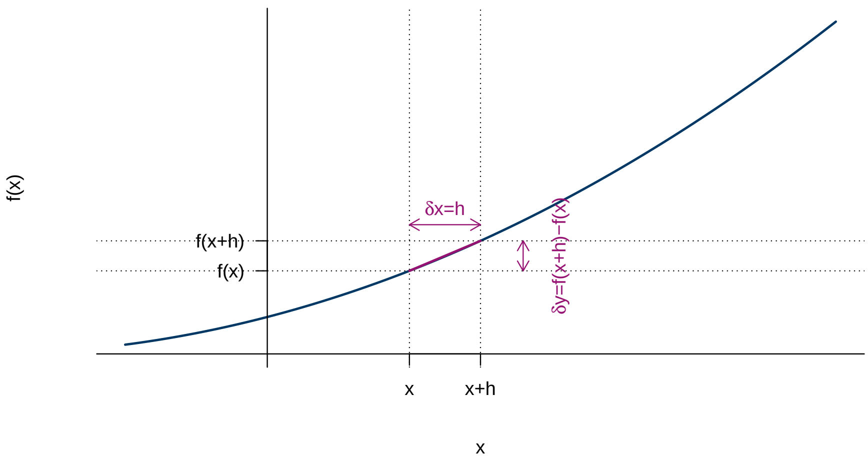
(a) $x = 1$ to $x = 4$
(b) $x = -2$ to $x = 0$

For the first interval the change in x is equal to $4 - 1 = 3$. When $x = 1$, $f(x) = 1$; while when $x = 4$, $f(x) = 16$. Thus, the change of $f(x)$ is $16 - 1 = 15$. So, the avarage rate of change across the interval $[1, 4]$ is $\frac{15}{3} = 5$. What does this mean though? It means that across the interval $[1, 4]$, on average the $f(x)$ value increases by 5 for every 1 unit increase in x.



This is a good time for you to try out the second interval. (The average rate of change turns out to be -2.)

Rate of change at a point We often need to know the rate of change of a function at a point, and not simply an average rate of change across an interval. Let's assume that b is really close to a. To better reflect this is our notation, we will call what we used to call a, x, and what we used to call b, $x + h$, with h being a very small number.



As mentioned earlier, the average rate of change of y across the interval $[x, x + h]$ is

$$\frac{\text{change in } y \text{ direction}}{\text{change in } x \text{ direction}} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

What do you think would happen if we assumed that the distance, h, between the two points was made increasingly small (in Mathematics notation $h \to 0$)?

If we assumed that, it would mean that the second point $x + h$ is really close to x. This is exactly what we will assume in order to find the rate of change at the point x. Let's say that we assumed that $h \to 0$. If we now focus again on the graph above and assume that $h \to 0$, the distance between the two points x and $x + h$ would get smaller and likewise the difference between their respective outputs, $f(x)$ and $f(x + h)$, would also get smaller. We can define those respective differences as δx and δy respectively. The term δx reads as "delta x" and represents a small change in the x direction. In our case $\delta x = x + h - x = h$ and $\delta y = f(x + h) - f(x)$.

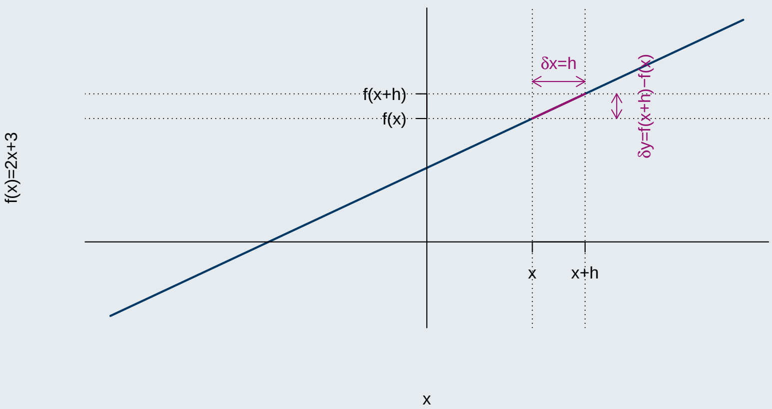
Thus, the **rate of change at a point** a is

$$\frac{\text{small change in } y \text{ direction}}{\text{small change in } x \text{ direction}} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Let's look at a couple of examples first and then focus on terminology and notation.

Example 2 (Rate of change at a point in a linear function).

One of the simplest functions to consider is a linear function. Let's assume that we have $f(x) = 2x + 3$.



What should we do if we want to find the rate of change at any point of the function? (We want to essentially answer the question "What is the change in the y direction when the change in the x direction is small")

Let's use the definition we saw earlier and calculate the rate of change at any point x of the function (think of it as looking at the two points x and $x + h$ with $h \to 0$).

$$\begin{aligned}\frac{\text{small change in } y \text{ direction}}{\text{small change in } x \text{ direction}} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\ &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{x+h-x} \\ &= \lim_{h \to 0} \frac{2(x+h)+3-(2x+3)}{x+h-x} \\ &= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}} \\ &= 2\end{aligned}$$

Wait. The rate of change for the function $f(x)$ at any point x is 2? What does that mean?

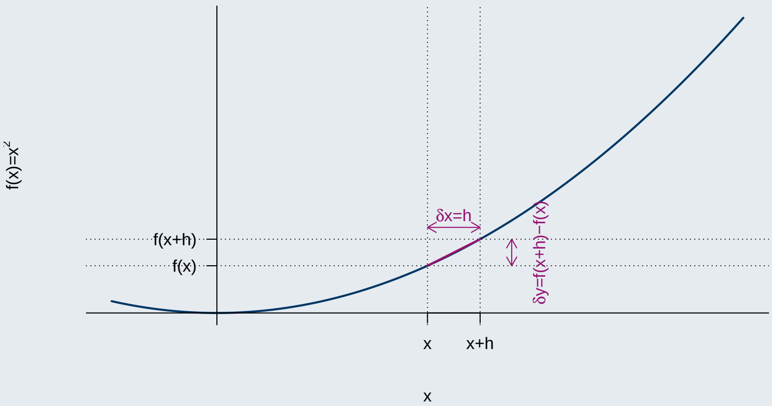
It means that the $f(x)$ value increases by $2h$ for every small increase, h, in x. So it doesn't matter which x value we are looking at (e.g. $x = 2$ or $x = 58.5$); the $f(x)$ value will always increase by 2 for every small increase, h, in x (i.e. $x = 2 + h$ or $x = 58.5 + h$ where $h \longrightarrow 0$).

For non-linear functions a one unit increase in the value of x leads to different increases in $f(x)$.

Example 3 (Quadratic function).

Consider a quadratic function $f(x) = x^2$.

Before we use the previous definition and calculate the rate of change at any point, let's try something else.



What will happen to the $f(x)$ values:

- if $x = 1$ and we increase it by 1 unit (i.e. $x = 2$)? The $f(x)$ values will increase by 3 (i.e. $2^2 - 1^2$).
- if $x = 2$ and we increase it by 1 unit (i.e. $x = 3$)? The $f(x)$ values will increase by 5 (i.e. $3^2 - 2^2$).
- if $x = 3$ and we increase it by 1 unit (i.e. $x = 4$)? The $f(x)$ values will increase by 7 (i.e. $4^2 - 3^2$).

Thus, in a quadratic function a 1 unit increase in x leads to different increases in the $f(x)$ values.

Let's now use the definition to find out what is happening in the $f(x)$ values when x is increased by h with $h \longrightarrow 0$ (instead of x being increased by 1).

$$\begin{aligned}
\frac{\text{small change in } y \text{ direction}}{\text{small change in } x \text{ direction}} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{x + h - x} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{x + h - x} \\
&= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{x + h - x} \\
&= \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{aligned}$$

So, the rate of change for the function $f(x)$ at a point x is $2x$. This means that the $f(x)$ value increases by $2x$ for every small increase, h, in x. Thus, the rate of change along a quadratic function is changing constantly (according to the value of x we are looking at), the rate of change has to be computed separately at each possible value of x. The rate of change is thus a local phenomenon: it does not give us any information about the rate of change globally.

Note that the rate of change, $2x$, for the function $f(x)$ is itself a function of x.

Terminology and notation The process of finding the rate of change of a given function is called **differentiation**. The function is said to be **differentiated**. If $f(x) \equiv y$ (read "$f(x)$ is equivalent to y") is a function of x we say that y is differentiated with respect to x. The rate of change of a function is also known as the **derivative** of the function.

There is a notation for writing down the derivative of a function. If the function is $f(x) \equiv y$, we denote the derivative of $y = f(x)$ by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(read "dee y (by) dee x", "dee f of x dee x" and "f prime").

This is the point where you should start asking yourselves "Wait a minute, do I have to compute $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ every time I need to find the derivative of a function at a point x?". Thankfully, the answer is no.

Table of derivatives Table 1 lists some of the common functions used in Mathematics and Statistics and their corresponding derivatives. The symbols k and n are constants while the symbol x represents a variable.

Function $f(x)$	Derivative $f'(x)$
constant	0
x	1
kx	k
x^n	nx^{n-1}
kx^n	knx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
a^x	$\log(a)a^x$
$\log(x)$	$\frac{1}{x}$
$\log(kx)$	$\frac{1}{x}$

Example 4 (Calculating derivatives of functions (1/15)).

Find the derivative of $f(x) = 3x$.

We note that $3x$ is of the form kx where $k = 3$. This means that $f'(x) = 3$.

Example 5 (Calculating derivatives of functions (2/15)).

Find the derivative of $f(x) = 3$.

This function is constant, hence its derivative is zero.

Example 6 (Calculating derivatives of functions (3/15)).

Find the derivative of $f(x) = 6x^2$.

This function is of the form kx^n with $k = 6$ and $n = 2$, hence its derivative is $12x$.

Example 7 (Calculating derivatives of functions (4/15)).

Find the derivative of $f(x) = \sqrt{x}$.

We first rewrite the function as $f(x) = \sqrt{x} = x^{\frac{1}{2}}$. This means that the function is of the form kx^n with

$k = 1$ and $n = \frac{1}{2}$. This means that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$.

Example 8 (Calculating derivatives of functions (5/15)).

Find the derivative of $f(x) = \frac{3}{x^2}$.

We first rewrite the function as $f(x) = \frac{3}{x^2} = 3x^{-2}$. This means that the function is of the form kx^n with $k = 3$ and $n = -2$. This means that $f'(x) = 3(-2)x^{-3} = -6x^{-3} = -\frac{6}{x^3}$.

Example 9 (Calculating derivatives of functions (6/15)).

Find the derivative of $f(x) = e^{3x}$.

This function is of the form e^{kx} with $k = 3$, hence its derivative is $f'(x) = 3e^{3x}$.

Ok, that is a good start but what do we do with functions like $f(x) = 2x + 3$, $g(x) = x^5 \log(x)$ and $h(x) = \frac{x^2}{e^x}$?

The first function involves adding two functions (the first one being of the form kx while the second one is a constant function).

The second function, $g(x)$, involves multiplying two functions (x^5 and $\log\{x\}$) while the last one, $h(x)$, involves dividing two functions (x^2 and e^x).

We need to introduce some simple rules to enable us to extend the range of functions that we can differentiate.

Rules of differentiation

- **Differentiation is linear:** For any functions f and g and any real numbers a and b, the derivative of the function $h(x) = af(x) \pm bg(x)$ with respect to x is $h'(x) = af'(x) \pm bg'(x)$
- **Product rule:** For any functions f and g the derivative of a function $h(x) = f(x)g(x)$ with respect to x is
$$h'(x) = f'(x)g(x) + f(x)g'(x).$$
- **Quotient rule:** For any functions f and g the derivative of a function $h(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$, with respect to x is
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$
- **Chain rule:** The derivative of the function of a composite function $h(x) = f(g(x))$ with respect to x is
$$h'(x) = f'(g(x))g'(x).$$
What is a composite function you ask? It is a function that takes another function as its argument. So, instead of having a function $f(x)$ that has x as its input, we have a function f which takes $g(x)$ as its input. Thus, it becomes $f(g(x))$.

Function $h(x)$	Derivative $h'(x)$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$af(x) - bg(x)$	$af'(x) - bg'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
$f(g(x))$	$f'(g(x))g'(x)$

Example 10 (Calculating derivatives of functions (7/15)).

Find the derivative of $h(x) = 4x^5 + 5x^2$.

This function is of the form $af(x) + bg(x)$ with $a = 4$, $f(x) = x^5$, $b = 5$ and $g(x) = x^2$. Hence, $f'(x) = 5x^4$ and $g'(x) = 2x$, which yields

$$h'(x) = af'(x) + bg'(x) = 4 \times 5x^4 + 5 \times 2x = 20x^4 + 10x$$

(We could have also used $a = 1$, $f(x) = 4x^5$, $b = 1$ and $g(x) = 5x^2$.)

Example 11 (Calculating derivatives of functions (8/15)).

Find the derivative of $h(x) = 3x - 6x^6$.

This function is of the form $af(x) - bg(x)$ with $a = 3$, $f(x) = x$, $b = 6$ and $g(x) = x^6$. Hence, $f'(x) = 1$ and $g'(x) = 6x^5$, which yields

$$h'(x) = af'(x) - bg'(x) = 3 \times 1 - 6 \times 6x^5 = 3 - 36x^5$$

Example 12 (Calculating derivatives of functions (9/15)).

Find the derivative of $h(x) = \frac{9}{x^2} + \frac{11}{x}$.

This function is of the form $af(x) + bg(x)$ with $a = 9$, $f(x) = x^{-2}$, $b = 11$ and $g(x) = x^{-1}$. Hence, $f'(x) = -2x^{-3}$ and $g'(x) = -x^{-2}$, which yields

$$h'(x) = af'(x) + bg'(x) = 9 \times (-2x^{-3}) + 11 \times (-x^{-2}) = -\frac{18}{x^3} - \frac{11}{x^2}$$

Example 13 (Calculating derivatives of functions (10/15)).

Find the derivative of $h(x) = xe^x$.

This function is of the form $f(x)g(x)$ with $f(x) = x$ and $g(x) = e^x$. Hence, $f'(x) = 1$, $g'(x) = e^x$ and

$$h'(x) = f'(x)g(x) + f(x)g'(x) = 1 \times e^x + x \times e^x = e^x + xe^x = (1 + x)e^x$$

Example 14 (Calculating derivatives of functions (11/15)).

Find the derivative of $h(x) = x \log(x)$.

This function is of the form $f(x)g(x)$ with $f(x) = x$ and $g(x) = \log(x)$. Hence, $f'(x) = 1$, $g'(x) = 1/x$ and

$$h'(x) = f'(x)g(x) + f(x)g'(x) = 1 \times \log(x) + x \times \frac{1}{x} = \log(x) + 1$$

Example 15 (Calculating derivatives of functions (12/15)).

Find the derivative of $h(x) = \frac{x^2-1}{x^3}$.

This function is of the form $\frac{f(x)}{g(x)}$ with $f(x) = x^2 - 1$ and $g(x) = x^3$. Hence, $f'(x) = 2x$, $g'(x) = 3x^2$ and

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{2x \times x^3 - (x^2 - 1) \times 3x^2}{(x^3)^2} \\ &= \frac{2x^4 - 3x^4 + 3x^2}{x^6} = \frac{x^2(-x^2 + 3)}{x^6} \\ &= \frac{3 - x^2}{x^4} \end{aligned}$$

Example 16 (Calculating derivatives of functions (13/15)).

Find the derivative of $h(x) = \frac{e^x + x}{x^2}$.

This function is of the form $\frac{f(x)}{g(x)}$ with $f(x) = e^x + x$ and $g(x) = x^2$. Hence, $f'(x) = e^x + 1$, $g'(x) = 2x$ and

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{(e^x + 1) \times x^2 - 2x \times (e^x + x)}{(x^2)^2} \\ &= \frac{x^2e^x + x^2 - 2xe^x - 2x^2}{x^4} = \frac{x^2e^x - x^2 - 2xe^x}{x^4} \\ &= \frac{xe^x - x - 2e^x}{x^3} \end{aligned}$$

Example 17 (Calculating derivatives of functions (14/15)).

Find the derivative of $h(x) = (x^3 + x)^7$.

We could first expand the power, but this would be extremely time-consuming. It is a lot easier to view $h(x) = f(g(x))$ with $f(x) = x^7$ and $g(x) = (x^3 + x)$.

The formula for the derivative of $h(x)$ is $h'(x) = f'(g(x))g'(x)$, i.e. we have to first differentiate both $f(x)$ and $g(x)$ yielding $f'(x) = 7x^6$ and $g'(x) = 3x^2 + 1$.

Next we need to find $f'(g(x))$, i.e. we have to use the value of $g(x)$ as the input to $f'(x)$. We do this by replacing every occurence of x in $f'(x)$ by $g(x) = x^3 + x$:

$$f'(g(x)) = 7(g(x))^6 = 7(x^3 + x)^6$$

Thus,

$$h'(x) = f'(g(x))g'(x) = 7(x^3 + x)^6\ (3x^2 + 1)$$

Example 18 (Calculating derivatives of functions (15/15)).

Find the derivative of $h(x) = \log(x^2 + x + 1)$.

We will again use the chain rule and write $h(x) = f(g(x))$ with $f(x) = \log(x)$ and $g(x) = x^2 + x + 1$. Then $f'(x) = \frac{1}{x}$ and $g'(x) = 2x + 1$ and thus

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) = \frac{1}{g(x)}g'(x) \\ &= \frac{1}{x^2 + x + 1} \times (2x + 1) = \frac{2x + 1}{x^2 + x + 1} \end{aligned}$$

Higher-order derivatives So far we have only looked at the first derivative. The derivative of $f'(x)$ is known as the **second derivative** and denoted as

$$f''(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = (f'(x))'.$$

While the first derivative contains information about the rate of change, which corresponds to the slope of a function, the second derivative contains information about the **curvature**.

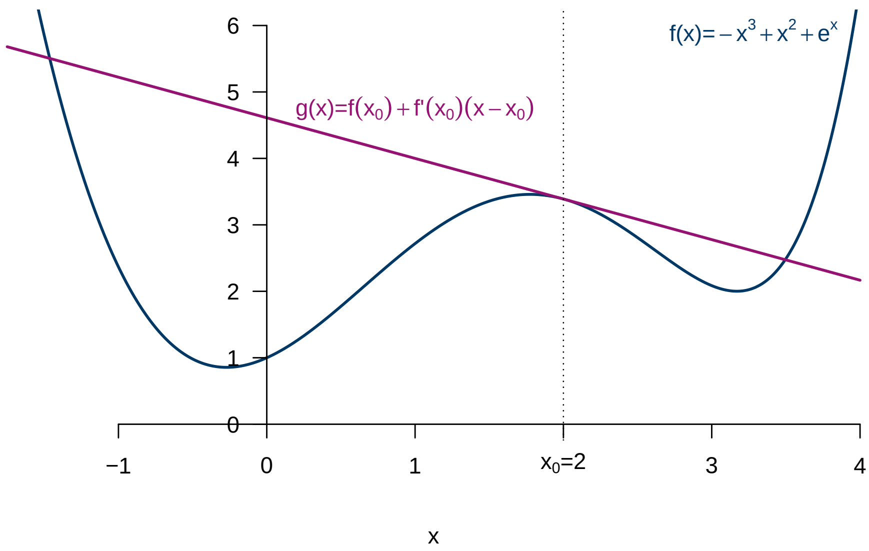
Example 19.

Find the first and second derivative of $f(x) = x^4 + 2x$.

The first derivative is $f'(x) = 4x^3 + 2$. To find the second derivative, we differentiate the first derivative once more.

$$f''(x) = (f'(x))' = (4x^3 + 2)' = 12x^2$$

Approximating functions using derivatives We can use derivatives to approximate functions. We have already seen that the derivative $f'(x_0)$ gives the slope of the function $f(x)$ at $x = x_0$.



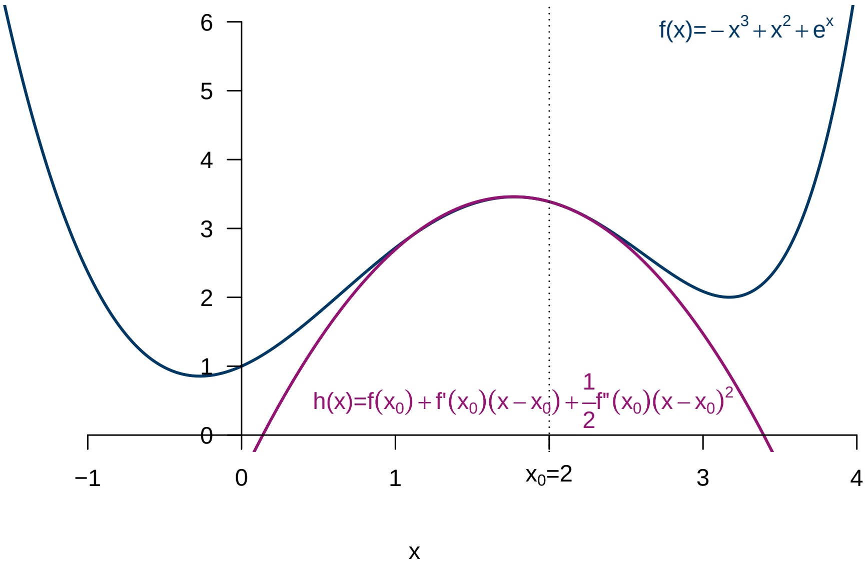
The line that corresponds to the slope at $x_0 = 2$ is actually a function itself. The purple line in the figure above is graph of the function

$$g(x) = f(x_0) + f'(x_0)(x - x_0).$$

This function provides a local approximation to the function $f(x)$ around $x = x_0$. The approximation $g(x)$ touches the function $f(x)$ at $x = x_0$ so that both functions take the same value and have the same slope at $x = x_0$.

Can we do a better job at approximating the function $f(x)$? The answer is yes: we can include a term involving the second derivative, so that we also match the curvature in $x = x_0$.

$$h(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x - x_0)(x - x_0)^2$$



We can keep adding higher-order derivatives in order to improve the approximation, which is known as Taylor series (or Maclaurin series) approximation.

Tasks

Task 1.

Find the derivative of $f(x) = x^3 + 8$?

Task 2.

Find the derivative of $f(x) = -\frac{1}{x^3}$?

Task 3.

Let f and g be functions that are differentiable everywhere.

Suppose that $f(2) = 3$, $g(2) = 2$, $f'(2) = -3$, and $g'(2) = -2$.

Use this information to determine the value of $h'(2)$, where $h = f(g(x))$.

Task 4.

The slope of the curve $f(x) = x^2 - 8x + 7$ is zero at which value(s) of x?

Task 5.

What is the derivative of $f(x) = \frac{e^x}{e^x+1}$?

Task 6.

What is the derivative of $f(x) = x^2 e^{3x}$?

Self help

Basic differentiation: a refresher

http://www.mathcentre.ac.uk/resources/Refresher%20Booklets/basic%20diff%20refresh1Emathcentre/final0203-ltsn-basicdiff.pdf

Understanding the definition of the derivative (YouTube video)

https://www.youtube.com/watch?v=2wH-g60EJ18

Lots of different derivative examples (YouTube video)

https://www.youtube.com/watch?v=ZvCWt4Bjbyl

Introduction to composite functions (Khan Academy video)

https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/v/function-composition

Finding composite functions (Khan Academy video)

https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/v/new-function-from-composition

Maximum and minimum values

Local and global extrema

The maximum and minimum values of a function are often very important. For example, we may want to know what value a parameter needs to take so that an algorithm performs best, as measured by an objective function.

It is important to distinguish between two types of maxima and minima.

- Local maxima (minima) are points at which the function takes larger (smaller) values than in its vicinity.
- Global maxima (minima) are points at which the function takes its largest (smallest) value.

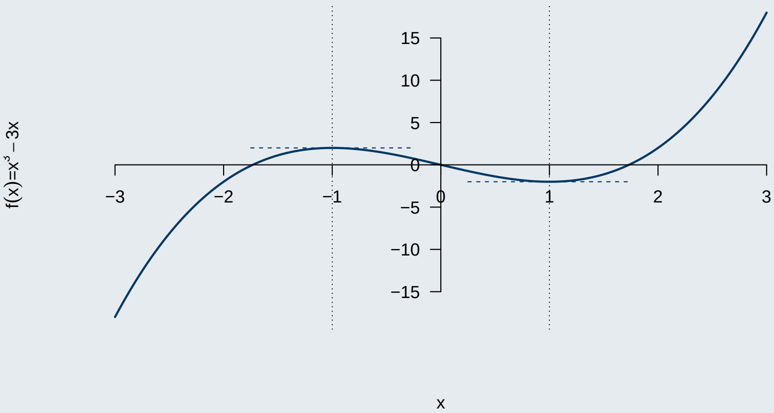
If it is clear from the context, local maxima (minima) are often just referred to as maxima (minima), without prefixing them by the word "local".

Differentiation and stationary points

Differentiation can be used to find the maximum and minimum values of a function. Since the derivative provides information about the slope (or gradient) of the graph of a function we can use it to locate points on a graph where the slope is zero. We will see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality.

Example 20 (Maximum and minimum of a cubic function).

Consider the function $f(x) = x^3 - 3x$. We may be interested in finding its maximum and minimum values.



We can see in the graph that the slope of the function is 0 in $x \approx -1$ and $x \approx 1$. Such points at which the slope to the graph is horizontal, thus zero, are called **stationary points**. You can also say that the rate of change of a function at stationary points is zero.

At the local maximum at $x \approx -1$ the function takes a larger value than in its vicinity. Note that this is not a global maximum, as the function takes larger values for large x (as $f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$). At the local minimum at $x \approx 1$ the function takes a smaller value than in its vicinity. Again, this is not a global minimum as the function takes smaller values for small x (as $f(x) \longrightarrow -\infty$ as $x \longrightarrow -\infty$).

You can probably notice by looking at the graph that the curve actually turns at the stationary points. As an example, let's focus on the local maximum at $x \approx -1$. We can see that the curve goes up right before it reaches the local maximum and then it goes down. The exact opposite happens at the local minimum. Thus, these stationary points are also referred to as **turning points**.

Drawing a graph of a function as above will reveal its behaviour, but if we want to know the precise location of such points we need to turn to algebra and differential calculus.

We have seen that the local maximum and local minimum are stationary points, i.e. we can find their exact location by solving the equation $f'(x) = 0$.

Solving

$$0 = f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$$

yields the roots $x = -1$ and $x = 1$, which correspond to the local maximum and minimum, respectively.

All turning points are stationary points; but not all stationary points are turning points. Can you draw a graph of a function that has a stationary point that is not a turning point? (**Hint**: Try to create a graph that around a specific point it has a slope of zero but the behaviour of the curve is the same around that point.)

Distinguishing between stationary points

Think about what happens to the slope of the graph from example 20 as we travel through the minimum turning point, from left to right, that is as x increases. To the left of the local minimum, right before $x = 1$, the slope is negative; then the slope becomes zero, and right after the minimum point the slope becomes positive. In other words, the slope $f'(x)$ is increasing as x increases. In other words, the second derivative $f''(x)$ is positive.

To summarise, if we want to find maximum or minimum values we can:

1. locate the position of stationary points, let's say x_1, x_2, by looking for points where $f'(x) = 0$, and
2. calculate the second derivative at those values (i.e. $f''(x_1)$, $f''(x_2)$).

If the second derivative is positive, then the stationary point is a minimum. If the second derivative is negative, then the stationary point is a maximum.

It is possible for second derivative (at a stationary point) to be equal to zero; in that case we do not have sufficient information about what kind of stationary point it is.

Tasks

Task 7.

Can you find the stationary points of the following functions and distinguish between them?

(a) $f(x) = x^2 - x$

(b) $f(x) = 2 + 3x - x^3$

(c) $f(x) = 3x^4 - 4x^3$

(d) $f(x) = x^4 - 2x^2 + 3$

Introduction to minimum and maximum points (Khan Academy video)

https://www.khanacademy.org/math/algebra/algebra-functions/maximum-and-minimum-points/v/relative-minima-maxima

Second derivative test (Khan Academy video)

https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-7/v/second-derivative-test

Exercises on second derivative test

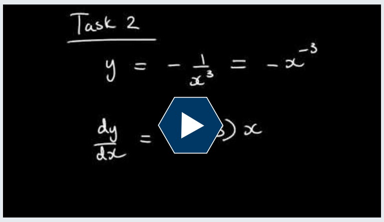
https://www.shmoop.com/second-derivatives/second-derivative-test-exercises.html

Answers to tasks

Answer to Task 1. We can differentiate the two terms independently of each other with the second term having a derivative of zero. Thus,

$$f'(x) = 3x^2$$

Answer to Task 2.



Video model answers

https://youtu.be/PN1NXtWtPsw

Duration: 0m39s

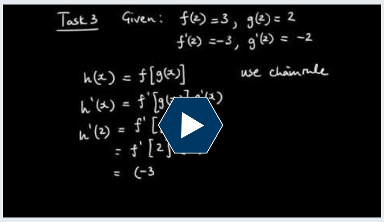
We first rewrite

$$f(x) = -\frac{1}{x^3} = -x^{-3}$$

and then take the derivative.

$$f'(x) = -(-3)x^{-4} = \frac{3}{x^4}$$

Answer to Task 3.



Video model answers

https://youtu.be/wrg0Y7LgqAU

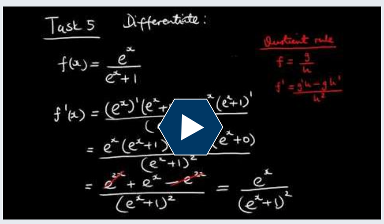
Duration: 1m20s

Using the chain rule for $h(x) = f(g(x))$ gives $h'(x) = f'(g(x))g'(x)$. Thus

$$h'(2) = f'(g(2))g'(2) = f'(2) \times (-2) = -3 \times (-2) = 6$$

Answer to Task 4. We first need to find $f'(x) = 2x - 8$. Setting $f'(x) = 0$ and solving for x yields $x = 4$.

Answer to Task 5.



Video model answers

https://youtu.be/LKlG2B2m4Ks

Duration: 2m00s

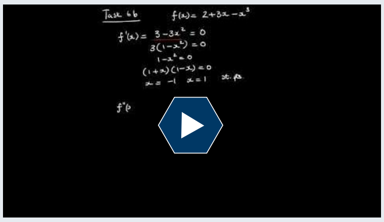
$f(x)$ is of the form $\frac{g(x)}{h(x)}$. Using the quotient rule gives the derivative $f'(x) = \frac{g'(x)h(x)-g(x)h'(x)}{h(x)^2}$.

$$\begin{aligned} f'(x) &= \frac{(e^x)'(e^x+1) - e^x(e^x+1)'}{(e^x+1)^2} \\ &= \frac{e^x(e^x+1) - e^x e^x}{(e^x+1)^2} \\ &= \frac{e^{2x}+e^x - e^{2x}}{(e^x+1)^2} \\ &= \frac{e^x}{(e^x+1)^2} \end{aligned}$$

Answer to Task 6. Using the product rule,

$$\begin{aligned} f(x) &= x^2 e^{3x} \\ f'(x) &= 2xe^{3x} + x^2 3e^{3x} \\ &= xe^{3x}(2+3x) \end{aligned}$$

Answer to Task 7.



Video model answers

https://youtu.be/Qyi1Xj0lYd8

Duration: 2m34s

(a) $f'(x) = 2x - 1$. Setting $f'(x) = 0$ yields $x = \frac{1}{2}$. $f''(x) = 2 > 0$, thus there is a local (and also global) minimum at $x = \frac{1}{2}$.

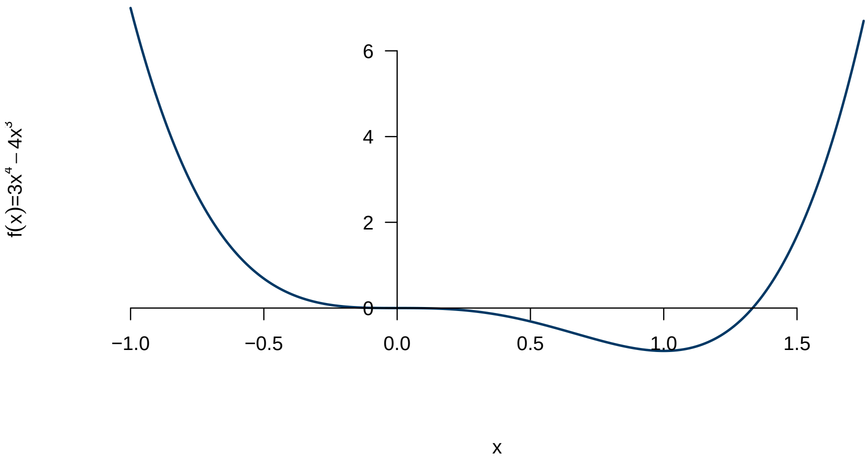
(b) $f'(x) = 3 - 3x^2 = 3(1-x^2) = 3(1+x)(1-x)$. As we have factorised the $f'(x)$ already, we know that it is zero for $x = -1$ and $x = 1$.

Taking the second derivative gives $f''(x) = -6x$. As $f''(-1) = 6 > 0$, there is a local minimum at $x = -1$. As $f''(1) = -6$, there is a local maximum at $x = 1$.

(c) $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$. $f'(x)$ is thus zero for $x = 0$ and $x = 1$.

$f''(x) = 36x^2 - 24x$, thus $f''(0) = 0$ and $f''(1) = 12 > 0$. Hence there is a local minimum at $x = 1$. We don't know yet about $x = 0$. Taking the third derivative gives $f'''(x) = 72x - 24$ and thus $f'''(0) = -24 \neq 0$, thus there is a saddle point at $x = 0$.

We can confirm this by plotting the function.



(d) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = x(x+1)(x-1)$, hence the derivative is zero for $x = -1$, $x = 0$ and $x = 1$.

The second derivative is $f''(x) = 12x^2 - 4$ yielding $f''(-1) = 8 > 0$ (local minimum at $x = -1$), $f''(0) = -4 < 0$ (local maximum at $x = 0$) and $f''(1) = 8 > 0$ (local minimum at $x = 1$).