Preliminary Mathematics for online MSc programmes in Data Analytics

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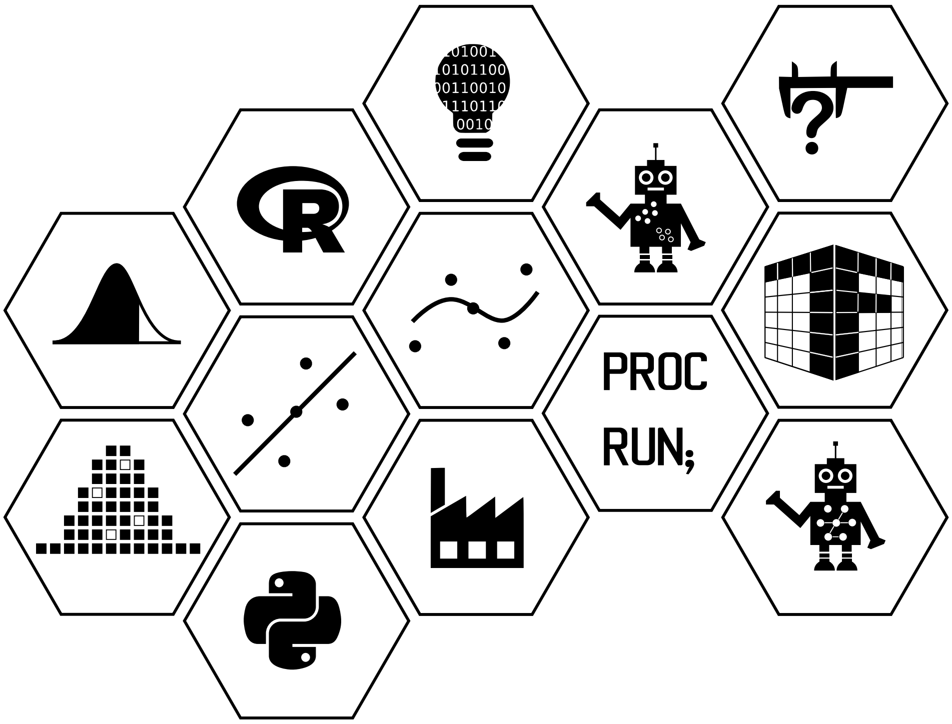
Unit 3:

Integration in 1D



Integration

Introduction to integration and area under the curve

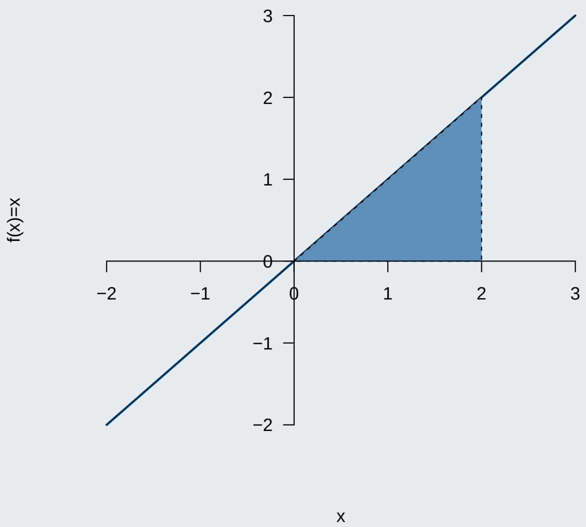
When a function $f(x) \equiv y$, is known, we can differentiate it to obtain the derivative $f'(x) \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$. The reverse process is to obtain $f(x)$ from knowledge of its derivative. This process is called **integration**.

However, integration is much more than just differentiation in reverse. It can be applied to finding areas under curves. By area under the curve we mean the area above the x-axis and below the graph of $f(x)$, assuming that $f(x)$ is positive. To give an example from Physics, the area underneath a graph of the velocity of an object against time represents the distance travelled by the object.

It is also important to regard integration as a process of adding up, or summation. Often, a physical quantity can be obtained by summing lots of small contributions or elements. For example, the position of the centre of mass of a solid body can be found by adding the contributions from all the small parts of which the body is composed.

Example 1 (Area under the curve).

Find the area under $f(x) = x$ between $x = 0$ and $x = 2$. You can see the graph of the function $f(x) = x$ below with the thick solid line and the area under the function with the grey triangle. The area forms a right triangle, with base and height both equal to 2. The area is therefore $\frac{1}{2} \cdot 2 \cdot 2 = 2$. More generally, the area underneath $f(x) = x$ between $x = 0$ and $x = a$ is a right triangle with base and height both equal to a. Thus, the area is $\frac{1}{2} \cdot a^2$.



Some of you may have already figured out that we just solved our first integral. Specifically, we calculated $\int_0^2 x\mathrm{d}x$ (and found out that the value of the integral is 2). Do not worry if you do not understand the previous sentence; I haven't explained the notation yet.

Terminology and notation The topic of integration can be approached in several different ways. Probably, the simplest way to think of it is as differentiation in reverse. In some applications we will know the derivative of the function, but not the function from which it was derived. This is why we need knowledge of integration.

Suppose we differentiate the function $f(x) \equiv y = x^2$ (meaning that $f(x)$ and y are equivalent; some times we will use one instead of the other). We will obtain $f'(x) \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. Integration reverses this process, and we say that the integral of $2x$ is x^2. Unfortunately, the situation we are in is actually a bit more complicated than that because there are lots of functions that we can differentiate to give $2x$. Here are some of them:

$$x^2 - 4, \quad x^2 + 25, \quad x^2 + \frac{1}{10}.$$

All these functions have the same derivative, $2x$, because when we differentiate the constant term we obtain zero. As a result, when we reverse the process, we have no idea what the original constant term might have been. Because of this we include in our answer an unknown constant, say c, called the **constant of integration**. We state that the integral of $2x$ is $x^2 + c$. There is nothing special about the letter c. We might use k (or something else), but we avoid using letters from the end of the alphabet like x, y, and z which are used for variables.

The symbol for integration is $\int$, known as an integral sign. Formally we write

$$\int 2x\mathrm{d}x = x^2 + c$$

where the term $2x$ within the integral is called the **integrand**, and the term $\mathrm{d}x$ indicates the name of the variable involved, in this case x.

We say that $2x$ is integrated with respect to x to give $x^2 + c$. Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals, which we will see (really) soon. When we find an indefinite integral our answer should always contain a constant of integration.

Can you now calculate the integral $\int 3x^2\mathrm{d}x$?

Table of integrals We could use a table of derivatives to find integrals, but the more common ones are usually found in a table of integrals such as the one seen in Table 1.

Function $f(x)$	Indefinite integral $\int f(x)\mathrm{d}x$
constant k	$kx + c$
kx	$k\frac{x^2}{2} + c$
kx^n ($n \neq 1$)	$k\frac{x^{n+1}}{n+1} + c$
e^x	$e^x + c$
e^{kx}	$\frac{e^{kx}}{k} + c$
$\frac{1}{x}$	$\log(x) + c$

Example 2 (Calculating simple indefinite integrals (1/3)).

What is $\int 4x^2\ \mathrm{d}x$?

The function $4x^2$ is of the form kx^n with $k = 4$ and $n = 2$, and thus

$$\int 4x^2\ \mathrm{d}x = 4\frac{x^3}{3} + c.$$

We can check whether we have calculated the integral correctly by differentiating the answer.

$$\left(4\frac{x^3}{3} + c\right)' = \frac{4}{3}(x^3)' = \frac{4}{3} \times 3x^2 = 4x^2,$$

which is the function we were meant to integrate.

Example 3 (Calculating simple indefinite integrals (2/3)).

What is $\int \frac{3}{x^2}\ \mathrm{d}x$?

We can rewrite the integrand as $\frac{3}{x^2} = 3x^{-2}$. It is thus of the form kx^n with $k = 3$ and $n = -2$. Thus,

$$\int \frac{3}{x^2}\ \mathrm{d}x = \int 3x^{-2}\ \mathrm{d}x = \frac{3x^{-1}}{-1} + c = -\frac{3}{x} + c$$

We can again check whether we have calculated the integral correctly by differentiating the answer.

$$\left(-\frac{3}{x} + c\right)' = (-3x^{-1})' = -3(-1)x^{-2} = 3x^{-2} = \frac{3}{x^2}$$

Example 4 (Calculating simple indefinite integrals (3/3)).

What is $\int e^{2x}\ \mathrm{d}x$?

The function e^{2x} is of the form e^{kx} with $k = 2$, and thus

$$\int e^{2x}\ \mathrm{d}x = \frac{e^{2x}}{2} + c.$$

We can again check whether we have calculated the integral correctly by differentiating the answer.

$$\left(\frac{e^{2x}}{2} + c\right)' = \frac{1}{2}(e^{2x})' = \frac{1}{2} \times 2e^{2x} = e^{2x}$$

Ok, that is a good start but what do we do with functions like $f(x) = 2x + 3x^2$ or $h(x) = x^2 e^x$?

The first function involves adding two functions (both of them being of the form kx^n).

The second function, $h(x)$, involves multiplication of two functions (x^2 and e^x).

We need to introduce some simple rules to enable us to extend the range of functions that we can integrate.

Rules of integration

- **Integration is linear:** For any functions f and g and any real numbers a and b the integral of the function

  $$h(x) = af(x) \pm bg(x)$$

  is

  $$\int h(x)\ \mathrm{d}x = \int af(x) \pm bg(x)\ \mathrm{d}x = a\int f(x)\ \mathrm{d}x \pm b\int g(x)\ \mathrm{d}x.$$

- **Substitution rule:** This rule involves making a substitution in order to simplify an integral and make it easier to calculate. For example, we may let a new variable, say u, equal a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience. Do not worry if at first you cannot see an appropriate substitution. However, it is not simply a matter of changing the variable, care must be taken with the term $\mathrm{d}x$ since the new integral will now include the variable u and not x.

- **Integration by parts:** The following formula

  $$\int f'(x)g(x)\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\mathrm{d}x$$

  shows us how we can calculate an integral of the form $\int f'(x)g(x)\mathrm{d}x$. We note that the previous formula replaces the original integral with a different integral, as seen on the right-hand side. This is because the second integral is easier to compute.

Function $h(x)$	Indefinite integral of $h'(x)$
$\int \big(af(x) + bg(x)\big)\mathrm{d}x$	$a\int f(x)\mathrm{d}x + b\int g(x)\mathrm{d}x$
$\int \big(af(x) - bg(x)\big)\mathrm{d}x$	$a\int f(x)\mathrm{d}x - b\int g(x)\mathrm{d}x$
$\int f'(x)g(x)\mathrm{d}x$	$f(x)g(x) - \int f(x)g'(x)\mathrm{d}x$

Example 5 (Calculating indefinite integrals of functions (1/5)).

For $h(x) = 2x + e^x$ calculate $\int h(x)\ \mathrm{d}x$

We note that $h(x)$ involves adding two functions (the first one is of the form kx^n while the second one is e^x). Thus, the integral of $h(x)$ is

$$
\begin{aligned}
\int h(x)\mathrm{d}x &= 2\int x\mathrm{d}x + \int e^x\mathrm{d}x \\
&= 2(\frac{x^2}{2} + c) + (e^x + c) \\
&= x^2 + 2c + e^x + c \\
&= x^2 + e^x + 3c \\
&\equiv x^2 + e^x + C
\end{aligned}
$$

In the final line, $3c$ has been replaced with a new constant C. It's just a tidier way of including a constant of integration in the final line of the calculated integral.

Example 6 (Calculating indefinite integrals of functions (2/5)).

For $h(x) = e^x x$ calculate $\int h(x)\ \mathrm{d}x$

We note that this involves multiplication of two functions, e^x and x. This example requires the use of the integration by parts rule.

Using this rule, we can calculate an integral that involves a product of two functions taking the form $\int f'(x)g(x)\mathrm{d}x$, with the first term being the derivative of $f(x)$ and the second one being $g(x)$. In this case, the two functions within the integral are e^x and x. If $f(x)$ and $g(x)$ are denoted as the functions e^x and x, we would be able to calculate the integral

$$
\int f'(x)g(x)\mathrm{d}x = \int (e^x)'x\mathrm{d}x = \int e^x x\mathrm{d}x.
$$

Now that we have $f(x) = e^x$ and $g(x) = x$, we can calculate the right-hand side of the integration by parts rule.

$$
\begin{aligned}
\int e^x x\ \mathrm{d}x &= \int f'(x)g(x)\mathrm{d}x \\
&= f(x)g(x) - \int f(x)g'(x)\mathrm{d}x \\
&= e^x x - \int e^x (x)'\mathrm{d}x \\
&= e^x x - \int e^x\mathrm{d}x \\
&= e^x x - (e^x + c) \\
&= e^x x - e^x - c \\
&\equiv e^x x - e^x + C
\end{aligned}
$$

Note that in the final line $-c$ has been replace with a new constant C.

Using integration by parts we were able to 'convert' a relatively complicated integral, $\int e^x x\mathrm{d}x$, into an 'easier' integral.

Example 7 (Calculating indefinite integrals of functions (3/5)).

For $h(x) = e^x x^2$ calculate $\int h(x)\ \mathrm{d}x$.

This is a similar integral to the previous example, except that we have x^2 to deal with instead of x. We will use integration by parts in this example as well.

Let $f'(x) = e^x$ and $g(x) = x^2$, then $f(x) = \int f'(x)\ \mathrm{d}x = e^x$ and $g'(x) = 2x$. Thus

$$
\begin{aligned}
\int h(x)\ \mathrm{d}x &= \int f'(x)g(x)\ dx \\
&= f(x)g(x) - \int f(x)g'(x)\ \mathrm{d}x \\
&= e^x x^2 - \int e^x \times 2x\ \mathrm{d}x \\
&= e^x x^2 - 2 \underbrace{\int e^x x\ \mathrm{d}x}_{= e^x x - e^x + c \text{ (from prev. ex.)}} \\
&= e^x x^2 - 2(e^x x - e^x + c) \\
&\equiv e^x (x^2 - 2x + 2) + C
\end{aligned}
$$

Example 8 (Calculating indefinite integrals of functions (4/5)).

For $h(x) = (3x+5)^6$ calculate $\int h(x)\ \mathrm{d}x$.

Here we have an example of a composite function. It is a function that raises $3x+5$ to the sixth power. We could expand the power to obtain a sum, and then integrate each term one-by-one, but this would be a lot of work.

We will substitute $3x+5$ with a new variable, say u. In that case, instead of having $(3x+5)^6$ we will have u^6. This gives us a much simpler function to integrate. However, there is a slight complication. The new function of u must be integrated with respect to u and not with respect to x. This means that we need to take care of the term $\mathrm{d}x$.

If we differentiate $u = 3x+5$ with respect to x we get:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}(3x+5)}{\mathrm{d}x} = (3x+5)' = 3.$$

It follows that we can write $\mathrm{d}x = \frac{1}{3}\mathrm{d}u$.

We now have that

$$\begin{aligned}\int (3x+5)^6\ \mathrm{d}x &= \int u^6 \frac{1}{3}\ \mathrm{d}u\\ &= \frac{1}{3}\int u^6\ \mathrm{d}u\\ &= \frac{1}{3}\Big(\frac{u^7}{7}+c\Big)\\ &= \frac{u^7}{21}+\frac{c}{3}\\ &\equiv \frac{u^7}{21}+c\end{aligned}$$

To finish off we must rewrite this answer in terms of the original variable x and we do this by replacing u with $3x+5$. So:

$$\int (3x+5)^6\ \mathrm{d}x = \frac{(3x+5)^7}{21}+c.$$

Example 9 (Calculating indefinite integrals of functions (5/5)).

For $h(x) = e^{2x-3}$ calculate $\int h(x)\ \mathrm{d}x$.

We have another composite function to integrate and to make our life easier, we will substitute $2x-3$ with u.

To rewrite the integral in terms of $\mathrm{d}u$ instead of $\mathrm{d}x$ we take the substitution $u = 2x-3$ and differentiate with respect to x, to get

$$\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}(2x-3)}{\mathrm{d}x} = (2x-3)' = 2,$$

thus $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.

By the way, we could have also rewritten $u = 2x-3$ as $x = \frac{1}{2}u+\frac{3}{2}$ and calculated

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{\mathrm{d}(\frac{1}{2}u+\frac{3}{2})}{\mathrm{d}u} = \left(\frac{1}{2}u+\frac{3}{2}\right)' = \frac{1}{2},$$

thus $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$, thus $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.

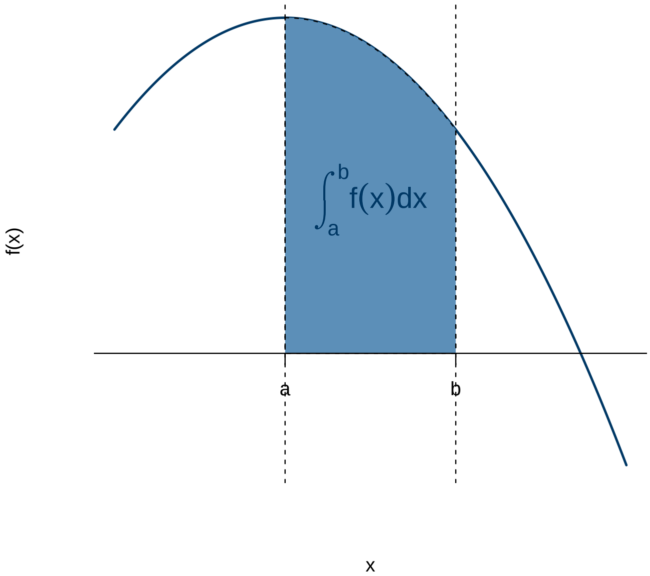
Either way, we can now write

$$\begin{aligned}
\int e^{2x-3}\,\mathrm{d}x &= \int e^{u}\,\frac{1}{2}\,\mathrm{d}u \\
&= \frac{1}{2}\int e^{u}\,\mathrm{d}u \\
&= \frac{1}{2}e^{u} + c \\
&= \frac{1}{2}e^{2x+3} + c
\end{aligned}$$

Definite integrals When integration was introduced as the reverse of differentiation the integrals we dealt with were indefinite integrals. The result of finding an indefinite integral is usually a function plus a constant of integration.

In this section, we introduce **definite integrals**, so called because the result will be a definite answer, usually a number, with no constant of integration. Definite integrals can be recognised by numbers written to the upper and lower right of the integral sign. The quantity $\displaystyle\int_a^b f(x)\mathrm{d}x$ is called the definite integral of $f(x)$ from a to b. The numbers a and b are known as the lower and upper limits of the integral respectively.

The definite integral $\int_a^b f(x)\,\mathrm{d}x$ is the (signed) area under the graph of f between a and b, as illustrated in the figure below.



These integrals have many applications (mentioned briefly in earlier sections), for example in finding areas bounded by curves or estimating probabilities. In the latter case $f(x)$ will not just be a function, it will be a density (you will see more of this in *Probability and Stochastic Models* or *Probability and Sampling Fundamentals*).

If the indefinite integral $\int f(x)\,\mathrm{d}x = F(x) + c$, i.e. $F'(x) = f(x)$, then the definite integral can be calculated as

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_{x=a}^{b} = F(b) - F(a)$$

When you evaluate a definite integral the result will usually be a number. Most importantly, when you integrate over a variable like x in a definite integral, the answer will never involve x. For an indefinite integral the answer would be likely to involve x.

Example 10 (Calculating definite integrals of functions (1/3)).

What is $\int_1^3 4x^2\,\mathrm{d}x$?

We have already found the indefinite integral in example 2:

$$\int 4x^2 \, \mathrm{d}x = 4\frac{x^3}{3} + c.$$

Thus,

$$\begin{aligned}\int_1^3 4x^2 \, \mathrm{d}x = \left[4\frac{x^3}{3}\right]_{x=1}^3 &= 4\frac{3^3}{3} - 4\frac{1^3}{3}\\ &= \frac{108}{3} - \frac{4}{3} = \frac{104}{3} \approx 34.67\end{aligned}$$

Example 11 (Calculating definite integrals of functions (2/3)).

What is $\int_1^4 (3x+5)^6 \mathrm{d}x$?

This is the same integral as the one we solved earlier in example 8 with the only difference being that now it is a definite integral. The procedure we have to follow is exactly the same as before with one small, but significant, addition. The limits of the new integral must be in terms of u and not in terms of x. This means that since $u = 3x + 5$, when $x = 1$ then $u = 8$; and when $x = 4$ then $u = 17$. Thus, we have:

$$\begin{aligned}\int_1^4 (3x+5)^6 \mathrm{d}x &= \int_8^{17} u^6 \frac{1}{3} \mathrm{d}u\\ &= \frac{1}{3}\int_8^{17} u^6 \mathrm{d}u\\ &= \frac{1}{3}\left[\frac{u^7}{7} + c\right]_{u=8}^{17}\\ &\equiv \frac{1}{3}\left(\left(\frac{17^7}{7} + c\right) - \left(\frac{8^7}{7} + c\right)\right)\\ &= \frac{1}{3}\left(\frac{17^7}{7} - \frac{8^7}{7}\right)\end{aligned}$$

You can stop right there, there is no need to calculate 17^7 and 8^7. Off the top of my head they should be close to 410338673 and 2097152 respectively.

Example 12 (Calculating definite integrals of functions (3/3)).

What is $\int_0^{+\infty} e^{-x} x \, \mathrm{d}x$?

This is a slightly more complicated example as we integrate all the way up to infinity.

We have already seen a similar indefinite integral in example 6 using integration by parts with $f'(x) = e^x$ and $g(x) = x$:

$$\int e^x x \, \mathrm{d}x = e^x x - e^x + c$$

This time the difference is that we have e^{-x} instead of e^x. Hence we will be using $f'(x) = e^{-x}$ and $g(x) = x$, yielding $f(x) = -e^{-x}$ and $g'(x) = 1$.

$$\int_0^{+\infty} e^{-x}x \, \mathrm{d}x = [-e^{-x}x]_{x=0}^{+\infty} - \int_0^{+\infty} -e^{-x} \times 1 \, \mathrm{d}x$$
$$= [-e^{-x}x]_{x=0}^{+\infty} - [e^{-x}]_{x=0}^{+\infty}$$
$$= \underbrace{-e^{-\infty}\infty}_{=0} - \underbrace{(-e^{-0}0)}_{=0} - \underbrace{e^{-\infty}}_{=0} + \underbrace{e^{-0}}_{=1}$$
$$= 1$$

You might feel that it is slightly dodgy notation to use infinity in a calculation as if it was a number. A cleaner way of writing $-e^{-\infty}\infty$ would be $\lim_{x\to+\infty} -e^{-x}x$. You might also be wondering why this quantity is 0. The reason is that, in the limit, e^{-x} decays faster to zero than any polynomial (including just x) could increase.

Tasks

Task 1.

If $\int_4^8 (2f(x) + 2) \, \mathrm{d}x = 16$, then what is $\int_4^8 f(x) \, \mathrm{d}x$?

Task 2.

Calculate the value of

$$\int_0^1 xe^x \, \mathrm{d}x$$

Task 3.

Evaluate

$$\int \sqrt{9y} \, \mathrm{d}y.$$

Task 4.

Evaluate

$$\int 2ye^{y^2} \mathrm{d}y.$$

Task 5.

Calculate the value of

$$\int_2^3 (3 - s)(1 + s) \, \mathrm{d}s.$$

Self help

Integration (Khan Academy material)

https://www.khanacademy.org/math/calculus-home/integration-calc

Exercises on integration by parts

http://www.cse.salford.ac.uk/physics/gsmcdonald/H-Tutorials/Integration-by-parts.pdf

Answers to tasks

Answer to Task 1. Using that itegration is a linear operator, we can rewrite

$$\int_4^8 (2f(x)+2)\mathrm{d}x = 2\int_4^8 f(x)\mathrm{d}x + \underbrace{\int_4^8 2\mathrm{d}x}_{=[2x]_{x=4}^8=2\times 8-2\times 4=16-8=8}$$
$$= 2\int_4^8 f(x)\mathrm{d}x + 8$$

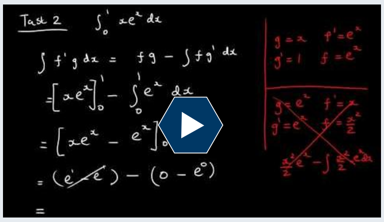
Thus

$$2\int_4^8 f(x)\mathrm{d}x + 8 = 16,$$

which we can re-arrange to

$$\int_4^8 f(x)\mathrm{d}x = \frac{16-8}{2} = 4$$

Answer to Task 2.



Video model answers

https://youtu.be/gr_e0rZ5eVY

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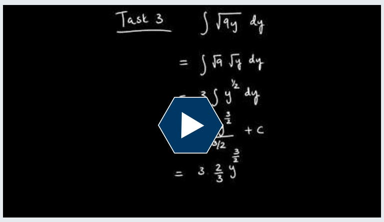
We use integration by parts ($\int_a^b f'(x)g(x)\ \mathrm{d}x = [f(x)g(x)]_{x=a}^b - \int_a^b f(x)g'(x)\ \mathrm{d}x$) with $f'(x) = e^x$ and $g(x) = x$. Hence $f(x) = e^x$ and $g'(x) = 1$.

$$\int_0^1 xe^x\ \mathrm{d}x = [e^x x]_{x=0}^1 - \int_0^1 e^x \times 1\mathrm{d}x$$
$$= [e^x x]_{x=0}^1 - [e^x]_{x=0}^1$$
$$= [e^x x - e^x]_{x=0}^1$$
$$= (e^1 \times 1 - e^0 \times 0) - (e^1 - e^0)$$
$$= e^0 = 1$$

Answer to Task 3.



Video model answers

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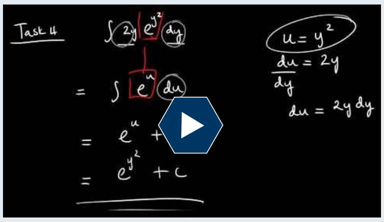
$$\int \sqrt{9y}\ \mathrm{d}y = \int \underbrace{\sqrt{9}}_{=3} \sqrt{y}\ \mathrm{d}y = 3\int y^{\frac{1}{2}}\ \mathrm{d}x$$
$$= 2y^{\frac{3}{2}} + c = 2\sqrt{y^3} + c$$

Answer to Task 4.



Video model answers

https://youtu.be/ViTzwZ6Hg8E

Duration: 1m51s

To calculate the integral

$$\int 2ye^{y^2}\ \mathrm{d}y$$

we will use the substitution $u = y^2$.

Differentiating $u = y^2$ gives $\frac{\mathrm{d}u}{\mathrm{d}y} = (y^2)' = 2y$, thus $\mathrm{d}u = 2y\mathrm{d}y$.

Thus

$$\begin{aligned}\int 2ye^{y^2}\ \mathrm{d}y &= \int e^u\ \mathrm{d}u \\ &= e^u + c = e^{y^2} + c\end{aligned}$$

Alternatively, we could have also solved for y and differentiated $y = \sqrt{u} = u^{\frac{1}{2}}$ giving $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$, thus $\mathrm{d}y = \frac{1}{2}u^{-\frac{1}{2}}\mathrm{d}u$, yielding

$$\begin{aligned}\int 2ye^{y^2}\ \mathrm{d}y &= \int 2u^{\frac{1}{2}}e^u\frac{1}{2}u^{-\frac{1}{2}}\ \mathrm{d}u \\ &= \int e^u\ \mathrm{d}u = e^u + c = e^{y^2} + c\end{aligned}$$

Answer to Task 5. The integral becomes easier to calculate if we first expand the integrand.

$$\begin{aligned}\int_2^3 (3-s)(1+s)\ \mathrm{d}s &= \int_2^3 3 + 3s - s - s^2\ \mathrm{d}s \\ &= \int_2^3 3 + 2s - s^2\ \mathrm{d}s \\ &= \left[3s + s^2 - \frac{1}{3}s^3\right]_{s=2}^3 \\ &= (3\times 3 + 3^2 - \frac{1}{3}3^3) - (3\times 2 + 2^2 - \frac{1}{3}2^3) = 9 - \frac{22}{3} \\ &= \frac{5}{3}\end{aligned}$$