Preliminary Mathematics for online MSc programmes in Data Analytics

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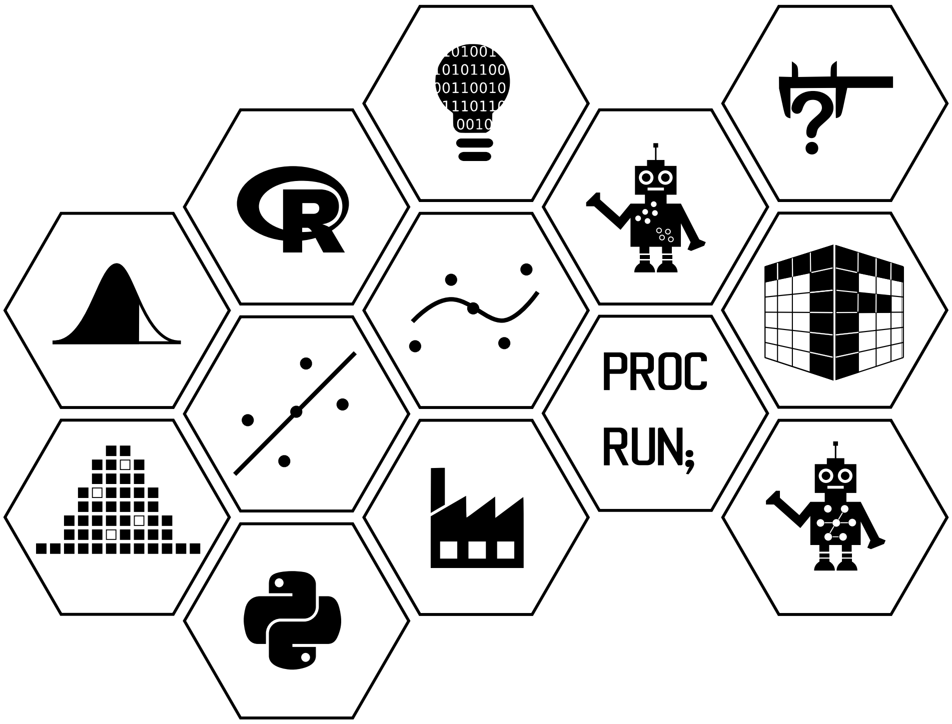
Academic Year 2023

Unit 4:

Functions of more than one variable and partial derivatives



DATA ANALYTICS GLASGOW

Functions of more than one variable

In Unit 1 we defined a function, $f(x)$, from a set A to another set B to be a rule that assigns for each $x \in A$ a unique element $f(x) \in B$. In that case the output $f(x)$ depended only on one input; the value of x.

However, we can also have two inputs (or more) to a function, each of which can be chosen independently. As before, the function can then be used to produce the output. Note that even though we can have two or more inputs, we will still assume that there is just a single output. These inputs can be represented by any letter (e.g. $x, y, z, \ldots$) and the output can either be represented by a letter (one that we have not used for any of the inputs) or as a function of the input (e.g. $f(x, y, z)$).

As an important special case, a function of two variables is a rule that produces a single output when specific values of the two variables are chosen. Similarly to the definition of Unit 1; a function, $f(x, y)$, from a set A to another set B defines a rule that assigns for each pair $(x, y) \in A$ a unique element $f(x, y) \in B$.

Example 1 (Function of two variables).

A function $f(x, y)$ is defined as $f(x, y) = \sqrt{x^2 + y^2 + 3x}$.

If we want to calculate its value for $x = 3$ and $y = 2$ we just need to find $f(3, 2) = \sqrt{3^2 + 2^2 + 3 \cdot 3} = \sqrt{22}$.

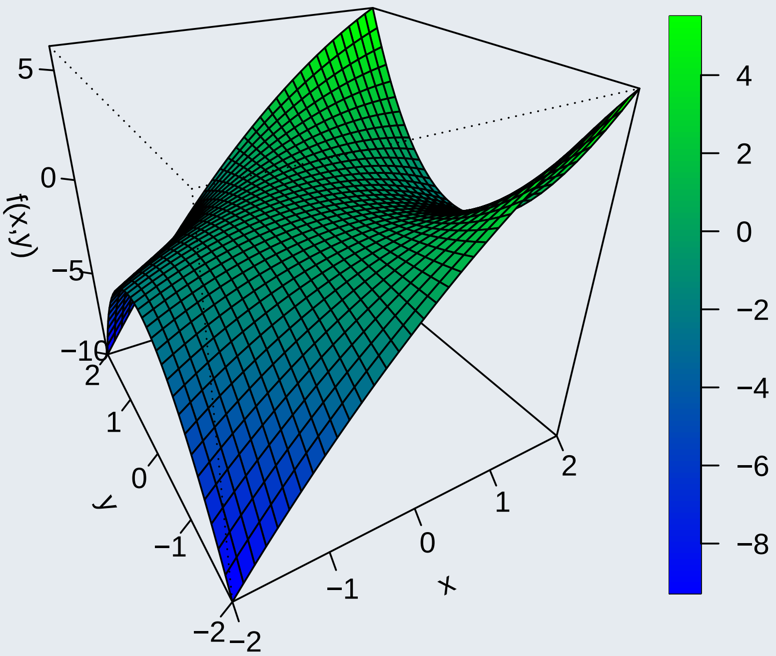
Similarly, if we want to calculate its value for $x = 2$ and $y = 3$ we need to find $f(2, 3) = \sqrt{2^2 + 3^2 + 3 \cdot 2} = \sqrt{19}$.

Example 2 (Graph of a function of two variables).

Let's assume that we now have a function $f(x, y)$ that has two inputs x, y:

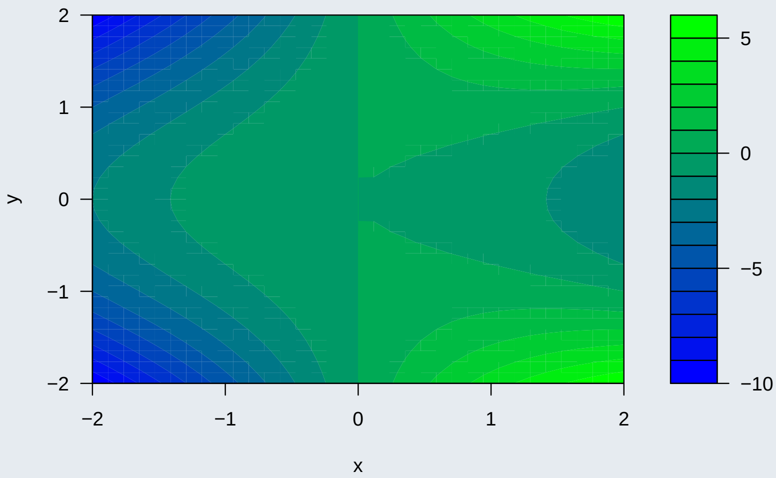
$$f(x, y) = -\frac{x^2}{2} + xy^2.$$

The figure below shows its graph for $-2 < x < 2$ and $-2 < y < 2$.



Because f is a function of x and y we can now ask questions like "How does $f(x, y)$ change as a function of x and y?" or "What happens to the function when x increases and y takes small values?". Would our answer be the same even if y was taking different values?

Another graphical technique for representing a 3-dimensional surface is by plotting constant f(x,y) slices, called contours, on a 2-dimensional format. That is, given a value for f(x,y), lines are drawn for connecting the (x,y) coordinates where that f(x,y) value occurs. Figure 2 shows the contour plot of $f(x, y)$.



Can you see the connection between the contour plot and the graph of the function (in the previous page)?

Focusing on the left half of the contour plot, we can see that we get exactly the same values at the top and the bottom of the contour plot (they both have the dark blue colour).

(**Hint**: What happens if I find the values of $f(x, y)$ for the two pairs $(x = -2, y = -2)$ and $(x = -2, y = 2)$ respectively?)

Partial differentiation

Introduction to partial differentiation

In Unit 2 we showed how one can differentiate a function of a single variable and compute the derivative of such a function. This gave us information about the slope of the graph of the function at different points. If x was the input and $f(x)$ (or y) was the output, we used the notation $f'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ to obtain the derivative. As a reminder, the latter way would be read as "the derivative of y with respect to the variable x".

Consider now that we have $f(x, y)$ (or z) as the function that depends on the two variables x and y. Thus, z can be differentiated with respect to x and produce one derivative while it can also be differentiated with respect to y and produce another (different) derivative. The derivative with respect to x gives us the slope in the x-direction, whereas the derivative with respect to y gives us the slope in the y-direction. So, for functions of two variables we can no longer talk about a single (unique) derivative of z.

From now on, when we differentiate a function of two variables, we will refer to this process as partial differentiation. Instead of using the letter d in $\frac{\mathrm{d}y}{\mathrm{d}x}$ we will use a curly d instead and write is as ∂. As an example, when we differentiate z with respect to x (or y), we will denote the resulting partial derivative as $\frac{\partial z}{\partial x}$ (or $\frac{\partial z}{\partial y}$). This is just a different notation, (almost) nothing else changes.

The rules for partial differentiation are the same as when we differentiate a function of a single input, with just one addition. When we differentiate with respect to a specific variable (let's say x) we treat all other variables as they were constants (i.e. numbers). Let's look at an example.

Example 3 (Partial differentiation (1/2)).

Let's assume we have the function

$$z = f(x, y) = 15x^2 + y.$$

If we differentiate with respect to x we should consider any occurence of all other variables as though they were constants. In this case, we only have one more variable y that we will consider as a constant.

Thus,

$$\frac{\partial z}{\partial x} = 30x$$

since the derivative of $15x^2$ is $30x$ while the derivative of the constant y is 0. If it helps, imagine that y had a specific value ($y = 20$) and the derivative of any number is 0.

If we differentiate with respect to y we should consider any occurence of all other variables (namely x) as though they were constants.

$$\frac{\partial z}{\partial y} = 1$$

since the derivative of $15x^2$ with respect to y is 0 and the derivative of y with respect to y is 1.

Example 4 (Partial differentiation (2/2)).

Let's now assume we have the function

$$z = f(x, y) = yxe^{2x}$$

and want to find the partial derivatives.

We start with the partial derivative with respect to x. For this we have to treat y as a constant and use the product rule.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} yxe^{2x} \\
&= y\frac{\partial}{\partial x} xe^{2x} \\
&= y(1 \times e^{2x} + x \times 2e^{2x}) = e^{2x}y(1+2x)
\end{aligned}$$

Note that we have used the product rule to differentiate xe^{2x} with respect to x.

The partial derivative with respect to y is a lot simpler to calculate.

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} yxe^{2x} \\
&= xe^{2x}\frac{\partial}{\partial y} y \\
&= xe2x \times 1 = xe^{2x}
\end{aligned}$$

Sometimes, we arrange the partial derivatives into a vector, called the **gradient** of the function. It is often denoted as $f'(x, y)$, $\nabla f(x, y)$, or $\frac{\mathrm{d}}{\mathrm{d}(x,y)}$.

$$f'(x, y) = \nabla f(x, y) = \frac{\mathrm{d}}{\mathrm{d}(x, y)} f(x, y) = \begin{bmatrix} \frac{\partial}{\partial x} f(x, y) \\ \frac{\partial}{\partial y} f(x, y) \end{bmatrix}$$

Higher-order derivatives

In the same way that a function of one variable has a second derivative (which is found by differentiating the first derivative), so too does a function of two variables. The second (order) partial derivatives are found by differentiating the first (order) partial derivatives. We can differentiate either of the first partial derivatives with respect to x or with respect to y to obtain various second partial derivatives.

- Differentiating $\frac{\partial z}{\partial x}$ with respect to x produces $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$.
- Differentiating $\frac{\partial z}{\partial x}$ with respect to y produces $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$.
- Differentiating $\frac{\partial z}{\partial y}$ with respect to x produces $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$.
- Differentiating $\frac{\partial z}{\partial y}$ with respect to y produces $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$.

In most circumstances (if the corresponding derivatives are continuous) the order of differentiation doesn't matter, in which case

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

The second derivatives are often arranged as a matrix, called the **Hessian**:

$$\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

Example 5 (Computing higher-order derivatives).

Let's assume we have the previous function $f(x, y) = 15x^2 + y$ and want to find the second partial derivatives.

Let's start with $\frac{\partial^2 z}{\partial x^2}$. We have already shown that $\frac{\partial z}{\partial x} = 30x$. If we differentiate again with respect to x we obtain 30. In more formal notation,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \underbrace{\frac{\partial}{\partial x}(15x^2 + y)}_{=30x} = \frac{\partial}{\partial x}(30x) = 30$$

Let's turn to $\frac{\partial^2 z}{\partial y \partial x}$. Derivatives are evaluated right-to-left, so we first have to differentiate with respect to x, just like before. This gave $\frac{\partial z}{\partial x} = 30x$. Now we have to take the derivative with respect to y. Given that there is no y in $30x$, the derivative with respect to y is 0. $\frac{\partial z}{\partial x} = 30x$.

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \underbrace{\frac{\partial}{\partial x}(15x^2 + y)}_{=30x} = \frac{\partial}{\partial y}(30x) = 0$$

Let's turn to $\frac{\partial^2 z}{\partial x \partial y}$, so we now swap the order of taking derivatives. We differentiate with respect to y first, giving $\frac{\partial z}{\partial y} = 1$, and differentiate the result with respect to x. As there is no x in the constant expression 1, the derivative is 0. More formally,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \underbrace{\frac{\partial}{\partial y}(15x^2 + y)}_{=1} = \frac{\partial}{\partial y}(1) = 0$$

We can see that, as expected, this is the same as

$$\frac{\partial^2 z}{\partial y \partial x}$$

.

Finally,

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \underbrace{\frac{\partial}{\partial y}(15x^2 + y)}_{=1} = \frac{\partial}{\partial y}(1) = 0.$$

Hence, the Hessian is

$$\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 30 & 0 \\ 0 & 0 \end{bmatrix}$$

Tasks

Task 1.

In each of the following cases, calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(a) $z = 5x + 12y$
(b) $z = 9 - 3y^4 + 12x^2$
(c) $z = 10(x + y + 5)$
(d) $z = 9x^2 y$
(e) $z = -9yx$

Task 2.

If $z = 11x + 2y^2$, evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(4, -3)$.

Task 3.

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(a) $z = e^x e^y$
(b) $z = e^{xy}$
(c) $z = e^{5x}$
(d) $z = e^{2y}$

Task 4.

In each of the following cases, calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

(a) $z = yxe^x$
(b) $z = 3xy^3e^x$
(c) $z = x \ln\{xy\}$
(d) $z = \frac{1}{x^2+y^2}$
(e) $z = \frac{x}{x^2+y^2}$

Task 5.

Find all the second partial derivatives in each of the following cases:

(a) $z = 8x + 2y + 11$
(b) $z = 10y^2x + 2$
(c) $z = -2x^4y^2$
(d) $z = 8e^{xy}$
(e) $z = \frac{1}{x}$
(f) $z = \frac{y}{x}$
(g) $z = \frac{x}{y}$

Self-help

Partial derivatives, introduction (Khan Academy video)

https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/partial-derivatives/v/partial-derivatives-introduction

Examples of partial derivatives (Part I)

http://personal.maths.surrey.ac.uk/st/S.Zelik/teach/calculus/partial_derivatives.pdf

Examples of partial derivatives (Part II)

http://tutorial.math.lamar.edu/Classes/CalcIII/PartialDerivatives.aspx

Stationary points of a function of two variables

Differentiation and stationary points

In Unit 2 we used differentiation to find the maximum and minimum values of a function of a single variable. You may ask yourselves "Is there a similar procedure for functions of two variables?" There is.

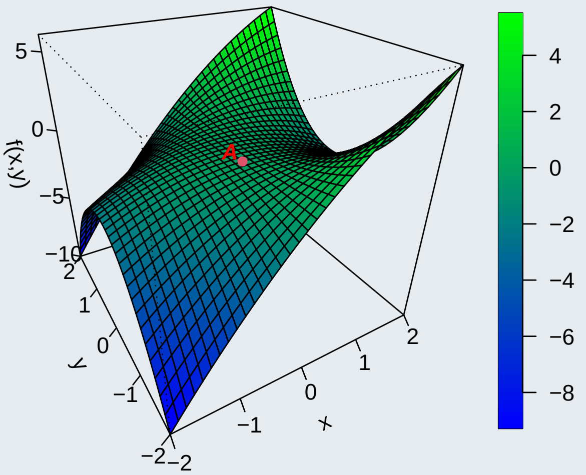
To locate stationary points in a function of 2 variables we have to set the first partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ equal to zero and find the values of x and y that satisfy this.

Example 6 (Finding stationary points of $f(x, y) = -x^2/2 + xy^2$).

Let's go back to the function $f(x, y) = -\frac{x^2}{2} + xy^2$ that we were looking at earlier. The first order partial derivatives are $\frac{\partial z}{\partial x} = -x + y^2$ and $\frac{\partial z}{\partial y} = 2xy$. We now need to find the values of x and y for which both partial derivatives are zero.

Setting the partial derivative with respect to x to zero yields $x = y^2$. We can now plug this into the derivative with respect to y (i.e. replace every occurrence of x by y^2), which yields $\left.\frac{\partial z}{\partial y}\right|_{x=y^2} = 2y^3$. Setting this to zero gives $y = 0$. Then $x = y^2 = 0$. Thus the only stationary point of the function is at $x = 0$ and $y = 0$.

The value of the function at that stationary point is the z coordinate and is obtained using $z = -\frac{0^2}{2} + 0 \cdot 0^2 = 0$. The graph of the function along with the stationary point A can be seen in the figure below.



Distinguishing between stationary points

So far, we have seen how to find the stationary points of a function but not how to distinguish between them. In order to do that we have to look at the second order derivatives.

Specifically, if we want to find the (local) maximum or (local) minimum values in a function of two variables we can:

1. locate the position of stationary points, let's say x_1, y_1, by looking for points where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are equal to zero, and
2. calculate the expression $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ at the stationary points x_1, y_1.

If the expression is:

- positive and $\frac{\partial^2 z}{\partial x^2}$ is positive we have a (local) minimum point,
- positive and $\frac{\partial^2 z}{\partial x^2}$ is negative we have a (local) maximum point,
- negative then we have what is known as a saddle point (a point that the slope is zero but its neither a minimum nor a maximum),
- zero the test is inconclusive and we need further tests to decide whether it is a minimum or a maximum.

Example 7 (Finding stationary points of $f(x,y) = -x^2/2 + xy^2$).

Let's determine whether the stationary point from example 6 is a (local) minimum, maximum or saddle point. We found the first partial derivatives in example 6.

$$\frac{\partial z}{\partial x} = -x + y^2 \quad \frac{\partial z}{\partial y} = 2xy$$

The second derivatives are

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-x + y^2\right) = -1 \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-x + y^2\right) = 2y \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(2xy) = 2x$$

We now need to evaluate

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -1 \times 2x - (2y)^2 = -2x - 4y^2$$

at $x = 0$ and $y = 0$, for which the quantity is 0. Our test is hence inconclusive.

From the figure in example 6 we can however see that it is a saddle point. (To see this from the derivatives we would have to take third-order partial derivatives).

Tasks

Task 6.

Locate the stationary points (and distinguish between them) of the following functions:

(a) $z = 3xy + x + y$
(b) $z = x^2 + y^2 - 3y$
(c) $z = x^2 + y^2 - 3xy$
(d) $z = \frac{1}{x} + \frac{1}{y} - \frac{3}{xy}$
(e) $z = -9yx$

Task 7.

Determine the stationary points of $f(x,y) = 2x^2 + 3y^2 + 5x + 12y + 19$.

Task 8.

Calculate $\frac{\partial z}{\partial x}$ when

(a) $z = \frac{y}{x^2} - \frac{x}{y^2}$
(b) $z = e^{x^2 - 4xy}$
(c) $z = \frac{x^2 - 3y^2}{x^2 + y^2}$

Self-Help

Local maximum and minimum values for a function of two variables (YouTube video)

https://www.youtube.com/watch?v=Hm5QnuDjNmY

Finding local maximums/minimums - second derivative test (YouTube video)

https://www.youtube.com/watch?v=QtXCIxB6kW8

Finding stationary points on functions of two variables (Part I)

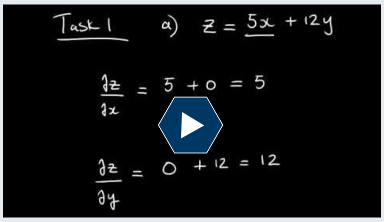
http://personal.maths.surrey.ac.uk/st/S.Zelik/teach/calculus/max_min_2var.pdf

Finding stationary points on functions of two variables (Part II)

https://archive.uea.ac.uk/jtm/14/dg14p10.pdf

Answers to tasks

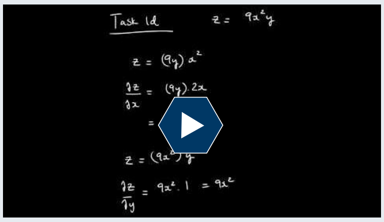
Answer to Task 1.



Video model answers for part (a)

https://youtu.be/5cMIlzLKC6M

Duration: 1m12s



Video model answers for part (d)

https://youtu.be/X1hWcg4VDyA

Duration: 1m02s

(a) $\frac{\partial z}{\partial x} = 5$; $\frac{\partial z}{\partial y} = 12$

(b) $\frac{\partial z}{\partial x} = 24x$; $\frac{\partial z}{\partial y} = -12y^3$

(c) $\frac{\partial z}{\partial x} = 10$; $\frac{\partial z}{\partial y} = 10$

(d) $\frac{\partial z}{\partial x} = 18xy$; $\frac{\partial z}{\partial y} = 9x^2$

(e) $\frac{\partial z}{\partial x} = -9y$; $\frac{\partial z}{\partial y} = -9x$

Answer to Task 2.

$$\frac{\partial z}{\partial x} = 11 \qquad \frac{\partial z}{\partial y} = 4y$$

Evaluating these partial derivatives at $x = 4$ and $y = -3$ gives

$$\left.\frac{\partial z}{\partial x}\right|_{x=4,y=-3} = 11 \qquad \left.\frac{\partial z}{\partial y}\right|_{x=4,y=-3} = 4 \times (-3) = -12$$

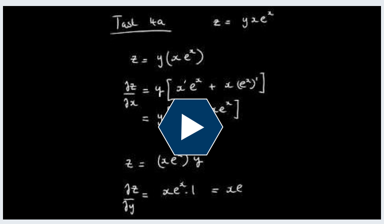
Answer to Task 3. (a) $\frac{\partial z}{\partial x} = e^x e^y$; $\frac{\partial z}{\partial y} = e^x e^y$

(b) $\frac{\partial z}{\partial x} = ye^{xy}$; $\frac{\partial z}{\partial y} = xe^{xy}$

(c) $\frac{\partial z}{\partial x} = 5e^{5x}$; $\frac{\partial z}{\partial y} = 0$

(d) $\frac{\partial z}{\partial x} = 0$; $\frac{\partial z}{\partial y} = 2e^{2y}$

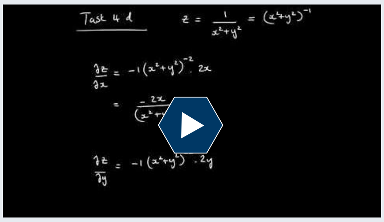
Answer to Task 4.



Video model answers for part (a)

https://youtu.be/PF0z_0Dbs4U

Duration: 1m35s



Video model answers for part (d)

https://youtu.be/3MHJDUUlWa4

Duration: 1m31s

(a) $\frac{\partial z}{\partial x} = ye^x + xye^x$; $\frac{\partial z}{\partial y} = xe^x$

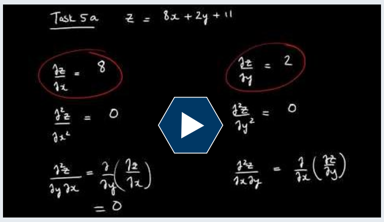
(b) $\frac{\partial z}{\partial x} = 3e^x y^3 + 3xe^x y^3$; $\frac{\partial z}{\partial y} = 9xy^2 e^x$

(c) $\frac{\partial z}{\partial x} = 1 + \ln(xy)$; $\frac{\partial z}{\partial y} = \frac{x}{y}$

(d) $\frac{\partial z}{\partial x} = -\frac{2x}{(x^2+y^2)^2}$; $\frac{\partial z}{\partial y} = -\frac{2y}{(x^2+y^2)^2}$

(e) $\frac{\partial z}{\partial x} = \frac{(y^2-x^2)}{(x^2+y^2)^2}$; $\frac{\partial z}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$

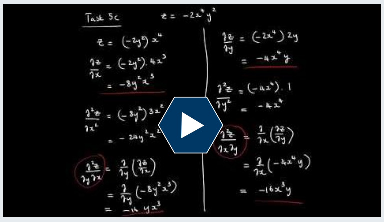
Answer to Task 5.



Video model answers for part (a)

https://youtu.be/lGGP1LW9RBI

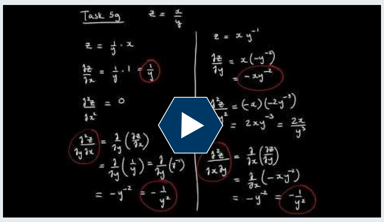
Duration: 1m26s



Video model answers for part (c)

https://youtu.be/s9XBcQwnTH4

Duration: 3m24s



Video model answers for part (g)

https://youtu.be/OCX0hFSDyJU

Duration: 3m31s

(a) $\frac{\partial^2 z}{\partial x^2} = 0$; $\frac{\partial^2 z}{\partial y^2} = 0$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 0$

(b) $\frac{\partial^2 z}{\partial x^2} = 0$; $\frac{\partial^2 z}{\partial y^2} = 20x$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 20y$

(c) $\frac{\partial^2 z}{\partial x^2} = -24x^2y^2$; $\frac{\partial^2 z}{\partial y^2} = -4x^4$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -16x^3 y$

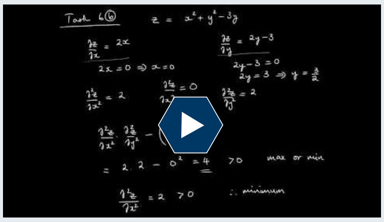
(d) $\frac{\partial^2 z}{\partial x^2} = 8y^2 e^{xy}$; $\frac{\partial^2 z}{\partial y^2} = 8x^2 e^{xy}$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 8[xye^{xy} + e^{xy}]$

(e) $\frac{\partial^2 z}{\partial x^2} = \frac{2}{x^3}$; $\frac{\partial^2 z}{\partial y^2} = 0$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 0$

(f) $\frac{\partial^2 z}{\partial x^2} = \frac{2y}{x^3}$; $\frac{\partial^2 z}{\partial y^2} = 0$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{x^2}$

(g) $\frac{\partial^2 z}{\partial x^2} = 0$; $\frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3}$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{y^2}$

Answer to Task 6.



Video model answers for part (b)

https://youtu.be/ZcLRPD2xAOc

Duration: 2m32s

(a) The partial derivatives are

$$\frac{\partial z}{\partial x} = 3y + 1 \quad \frac{\partial z}{\partial y} = 3x + 1$$

Setting both partial derivatives to zero yields $x = -\frac{1}{3}$ and $y = -\frac{1}{3}$.

The second derivatives are

$$\frac{\partial^2 z}{\partial x^2} = 0 \quad \frac{\partial^2 z}{\partial y \partial x} = 3 \quad \frac{\partial^2 z}{\partial y^2} = 0$$

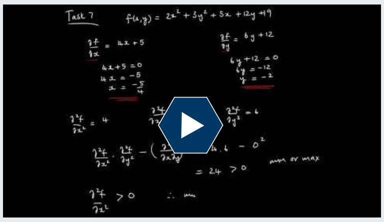
Thus

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0 \times 0 - 3^2 = -9 < 0,$$

hence there is a saddle point at $x = -\frac{1}{3}$ and $y = -\frac{1}{3}$.

(b) $x = 0$, $y = \frac{3}{2}$; minimum
(c) $x = 0$, $y = 0$; saddle point
(d) $x = 3$, $y = 3$; saddle point
(e) $x = 0$, $y = 0$; saddle point

Answer to Task 7.



Video model answers

https://youtu.be/LZaJXDBi1PI

Duration: 2m24s

The stationary point is at $x = -\frac{5}{4}$, $y = -2$. It is a minimum.

Answer to Task 8. (a) $\frac{\partial z}{\partial x} = -\frac{2y}{x^3} - \frac{1}{y^2}$

(b) $\frac{\partial z}{\partial x} = (2x - 4y)e^{x^2 - 4xy}$

(c) $\frac{\partial z}{\partial x} = \frac{8xy^2}{(x^2+y^2)^2}$