University of Glasgow

Preliminary Mathematics for online MSc programmes in Data Analytics

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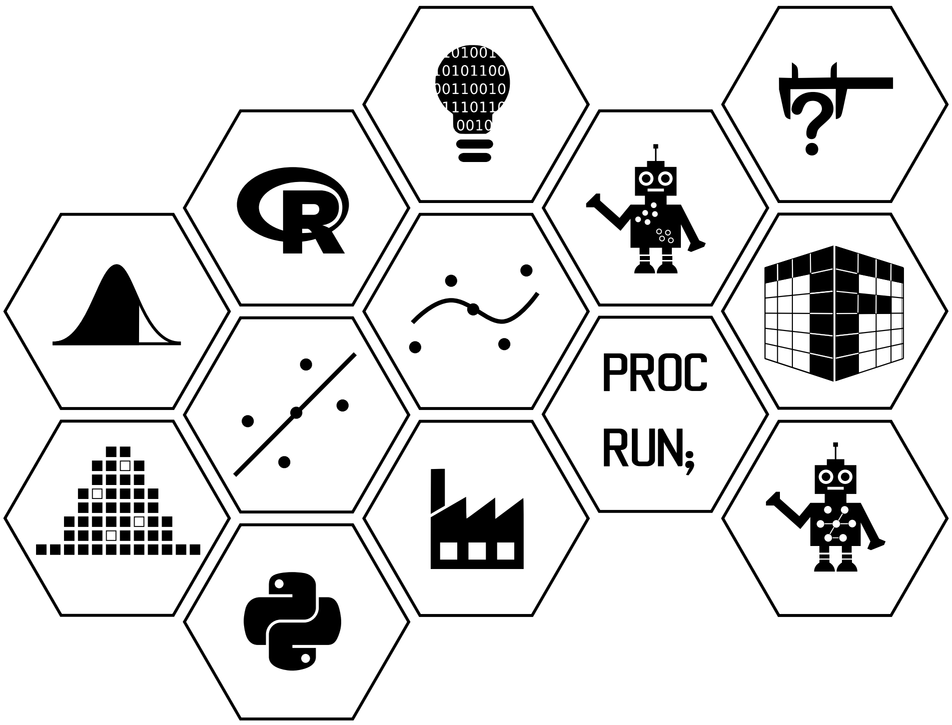
Unit 5:

Integration in higher dimensions

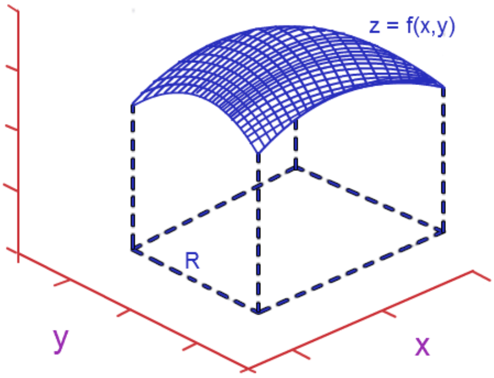


DATA ANALYTICS GLASGOW

Integration in higher dimensions

Introduction to integration in higher dimensions and volume of the region

In this Unit we will extend the idea of a definite integral, seen in Unit 3, to double integrals of functions of two variables. The definite integral of a function of one variable represents the area under the curve. Similarly, the double integral of a function of two variables represents the volume of the region between the surface defined by the function and the plane which contains its domain. This can also be used to calculate probabilities when two random variables are involved.



The figure above shows the graph of a function $f(x, y)$ over its domain R where R can be any rectangular region $R = [a, b] \times [c, d]$. This notation means that the first variable, x, of the function takes values between a and b while the second variable, y, takes values between c and d. In this case the double integral can be denoted as

$$\int\int_R f(x, y)\mathrm{d}x\mathrm{d}y = \int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y.$$

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then we can reverse the order of integration. This means that

$$\int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y = \int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x.$$

Notice that the inner differential matches up with the limits on the inner integral and similarly for the outer differential and limits. In other words, if the inner differential is $\mathrm{d}y$ then the limits on the inner integral must be y limits of integration and if the outer differential is $\mathrm{d}y$ then the limits on the outer integral must be y limits of integration.

In the above example, the integration over x goes from a to b and the integration over y goes from c to d.

Partial integration Let's look at the inner integral, $\int_c^d f(x, y)\mathrm{d}y$, of the last equation. We use the notation $\int_c^d f(x, y)\mathrm{d}y$ to mean that x is held fixed and we integrate the function $f(x, y)$ with respect to y from $y = c$ to $y = d$. This procedure is called **partial integration** with respect to y.

Now, $\int_c^d f(x, y)\mathrm{d}y$ is a number that depends on the value of x, so it defines a function of x such as $A(x) : \int_c^d f(x, y)\mathrm{d}y$. If we now integrate the function $A(x)$ with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x)\mathrm{d}x = \int_a^b \Big[\int_c^d f(x, y)\mathrm{d}y\Big]\mathrm{d}x.$$

The integral on the right side of the equation is called an **iterated integral** and the brackets are usually omitted. Thus

$$\int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x = \int_a^b \Big[\int_c^d f(x, y)\mathrm{d}y\Big]\mathrm{d}x$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

In a similar fashion, the iterated integral

$$\int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y = \int_c^d \Big[\int_a^b f(x, y)\mathrm{d}x\Big]\mathrm{d}y$$

means that we first integrate with respect to x from a to b and then with respect to y from c to d.
Notice that in both equations we work from inside out.

Example 1 (Evaluating an iterated integral).

Let's assume that we have the function $f(x, y) = x^2 y$ and we want to evaluate its volume over the region $R = [0, 3] \times [1, 2]$.

This means that we want to evaluate the integral

$$\int_0^3 \int_1^2 x^2 y \mathrm{d}y \mathrm{d}x.$$

As we previously mentioned, we can start by focusing on the inside integral $\int_1^2 x^2 y \mathrm{d}y$ and regard x as a constant. In that case, we will have

$$\begin{aligned} \int_1^2 x^2 y \mathrm{d}y &= x^2 \int_1^2 y \mathrm{d}y \\ &= x^2 \left[\frac{y^2}{2}\right]_{y=1}^{y=2} \\ &= x^2 \left(\frac{2^2}{2} - \frac{1^2}{2}\right) \\ &= \frac{3x^2}{2} \end{aligned}$$

Thus, the function $A(x)$ in the preceding discussion is equal to $\frac{3x^2}{2}$.
The only thing left to do right now is to integrate this function with respect to x from $x = 0$ to $x = 3$.

$$\begin{aligned} \int_0^3 \int_1^2 x^2 y \mathrm{d}y \mathrm{d}x &= \int_0^3 \left[\int_1^2 x^2 y \mathrm{d}y\right] \mathrm{d}x \\ &= \int_0^3 \frac{3x^2}{2} \mathrm{d}x \\ &= \frac{3}{2} \int_0^3 x^2 \mathrm{d}x \\ &= \frac{3}{2} \left[\frac{x^3}{3}\right]_{x=0}^{x=3} \\ &= \frac{3}{2} \left(\frac{3^3}{3} - \frac{0^3}{3}\right) \\ &= \frac{27}{2}. \end{aligned}$$

That's it. Can you reverse the order of integration and solve the integral, i.e.

$$\int_1^2 \int_0^3 x^2 y \mathrm{d}x \mathrm{d}y$$

Tasks

Task 1.

Evaluate the double integral $\int\int_R (x - 3y^2) \mathrm{d}x \mathrm{d}y$ where $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$

Task 2.

Find $\int_0^5 f(x, y) \mathrm{d}x$ and $\int_0^1 f(x, y) \mathrm{d}y$ for

(a) $f(x, y) = 12x^2 y^3$

(b) $f(x, y) = y + xe^y$

Task 3.

Calculate the iterated integrals:

(a) $\int_1^4 \int_0^2 (6x^2y - 2x)\mathrm{d}y\mathrm{d}x$
(b) $\int_1^4 \int_1^2 (\frac{x}{y} + \frac{y}{x})\mathrm{d}y\mathrm{d}x$
(c) $\int_0^1 \int_0^1 w(z + w^2)^4\mathrm{d}z\mathrm{d}w$

Task 4.

Calculate the following double integrals:

(a) $\int\int_R \frac{xy^2}{x^2+1}\mathrm{d}x\mathrm{d}y$ where $R = \{(x, y) \mid 0 \leq x \leq 1, -3 \leq y \leq 3\}$
(b) $\int\int_R ye^{-xy}\mathrm{d}x\mathrm{d}y$ where $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$

Self help

Double integrals (Khan Academy video)

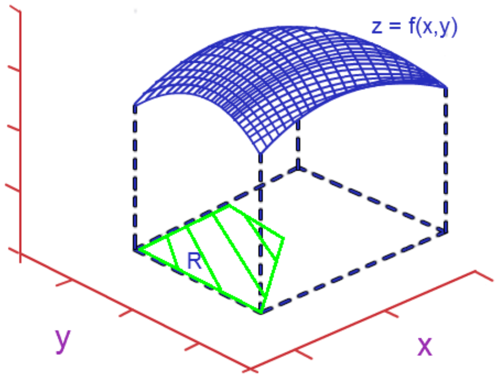
https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integral-1

Examples of iterated integrals

http://math.etsu.edu/multicalc/prealpha/Chap4/Chap4-1/printversion.pdf

Integration over general regions

For single integrals, the regions over which we integrate is always an interval. In the previous section we looked at double integrals which have to be integrated over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral



$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_R f(x,y)\mathrm{d}y\mathrm{d}x$$

where R is a non-rectangular region. The figure above shows the graph of a function $f(x,y)$ over its domain R where R is now the non-rectangular region in green. These integrals are unfortunately more complex to solve compared to the integrals over rectangular regions. The reason being it could be the case that the left hand side integral, on the previous equation, is difficult to solve while the one on the right-hand side is easy. This means that we would have to change the order of integration and put the correct limits of integration for the new integral.

Changing the order of integration is slightly tricky because its hard to write down a specific algorithm for the procedure. The easiest way to accomplish the task is through drawing a picture of the region R. From the picture, we can determine the corners and edges of the region R, which is what we need to work out (i.e. the limits of integration). Let's look at an example.

Example 2 (Evaluating a double integral over a non-rectangular region).

Assume that we want to calculate the integral

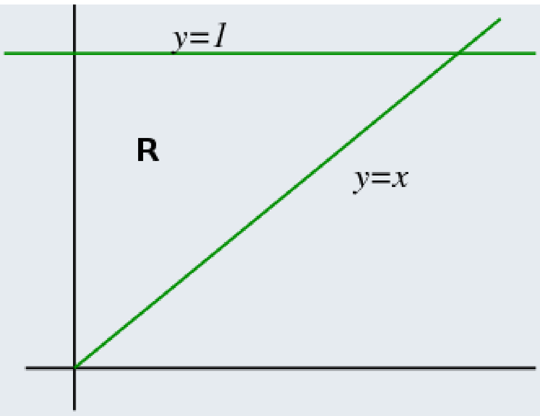
$$\int_0^1\int_x^1 e^{y^2}\mathrm{d}y\mathrm{d}x.$$

Evaluating the inner integral with respect to y will be very difficult as there is no anti-derivative of e^{y^2} (i.e. a function that when we differentiate it we will get e^{y^2}). But, if we manage to change the order of integration, we can integrate with respect to x first which is doable. And, it turns out that the integral with respect to y also becomes possible after we finish integrating with respect to x.

According to the limits of the integral, the region R can be described as

$$0 \le x \le 1$$
$$x \le y \le 1$$

The figure below shows the aforementioned region.



Since we can also describe the region R by

$$0 \le y \le 1$$
$$0 \le x \le y$$

the integral with the order changed is

$$\int_0^1 \int_0^y e^{y^2} \mathrm{d}x\mathrm{d}y.$$

Is it easier to solve this integral instead of the one we started with? Let's check this by starting with the inner integral first and integrate with respect to x (ending up with a function $A(y)$).

$$\begin{aligned}\int_0^y e^{y^2} \mathrm{d}x &= e^{y^2} \int_0^y 1\mathrm{d}x \\ &= e^{y^2} \left[x\right]_{x=0}^{x=y} \\ &= e^{y^2} \left(y - 0\right) \\ &= e^{y^2} y\end{aligned}$$

Thus, the function $A(y) = e^{y^2} y$.

The only thing left to do right now is to integrate the function with respect to y from $y = 0$ to $y = 1$.

$$\begin{aligned}\int_0^1 \int_0^y e^{y^2} \mathrm{d}x\mathrm{d}y &= \int_0^1 \left[\int_0^y e^{y^2} \mathrm{d}x\right] \mathrm{d}y \\ &= \int_0^1 e^{y^2} y\mathrm{d}y\end{aligned}$$

This can be integrated by using the substitution $u = y^2$. Using this substitution we will also have $\mathrm{d}u = 2y\mathrm{d}y$. We also need to work out the limits of the new integral. When $y = 0$ we have $u = 0^2 = 0$ and when $y = 1$ we get $u = 1^2 = 1$. We now have

$$\begin{aligned}\int_0^1 e^{y^2} y\mathrm{d}y &= \int_0^1 e^u \frac{\cancel{y}}{2\cancel{y}}\mathrm{d}u \\ &= \int_0^1 e^u \frac{1}{2}\mathrm{d}u \\ &= \frac{1}{2}\int_0^1 e^u \mathrm{d}u \\ &= \frac{1}{2}\left[e^u\right]_{u=0}^{u=1} \\ &= \frac{1}{2}\left(e^1 - e^0\right) \\ &= \frac{1}{2}\left(e-1\right)\end{aligned}$$

Example 3 (Evaluating a double integral over a non-rectangular region).

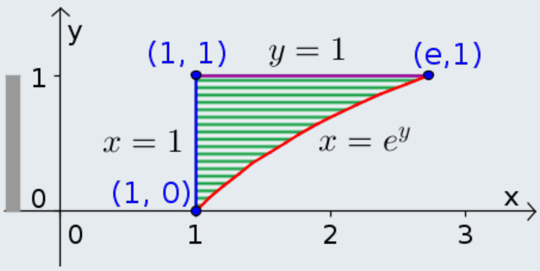
The focus of this example is on the limits of integration so there is no need to specify the function $f(x, y)$. The procedure does not depend on the identity of f.

Assume that we have a function $f(x, y)$ and we want to calculate the integral

$$\int_0^1 \int_1^{e^y} f(x, y)\mathrm{d}x\mathrm{d}y.$$

We can see that on the previous integral, the integration order is $\mathrm{d}x\mathrm{d}y$. As we previously mentioned, this corresponds to first integrating with respect to x from $x = 0$ to $x = e^y$, and afterwards integrating with respect to y from $y = 0$ to $y = 1$. Our task is to change the order of integration to be $\mathrm{d}y\mathrm{d}x$.

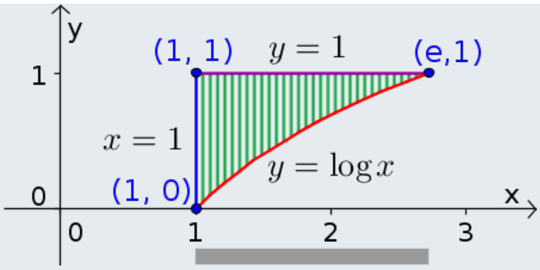
We begin by transforming the limits of integration into the domain R. The limits of the outer $\mathrm{d}y$ integral mean that $0 \leq y \leq 1$, and the limits on the inner $\mathrm{d}x$ integral mean that for each value of y the range of x is $1 \leq x \leq e^y$. The region R is shown in the figure below.



The range of y over the region is from 0 to 1, as indicated by the gray bar to the left of the figure. The horizontal hashing within the figure indicates the range of x for each value of y, beginning at the left edge $x = 1$ (blue line) and ending at the right curve edge $x = e^y$ (red curve).

We have also labelled all the corners of the region. The upper-right corner is the intersection of the line $y = 1$ with the curve $x = e^y$. Therefore, the value of x at this corner must be $e^1 = e$, and the point is $(e, 1)$.

To change the order of integration, we need to write an integral with order $\mathrm{d}y\mathrm{d}x$. This means that x should be the variable of the outer integral. Its limits must be constant and correspond to the total range of x over the region R. The range of x is $1 \leq x \leq e^1$, as indicated by the gray bar below the region in the figure below.



Since y will be the variable for the inner integration, we need to integrate with respect to y first. The vertical hashing indicates how, for each value of x, we will integrate from the lower boundary (red curve) to the upper boundary (purple line). These two boundaries determine the range of y. Since we can rewrite the equation $x = e^y$ for the red curve as $y = \log\{x\}$, the range of y is $\log\{x\} \leq y \leq 1$. (Note that this indicates the natural logarithm since the base of this logarithmic function is e. This means that we can write it as $\ln\{x\}$ instead)

In summary, the region R can be described not only by

$$\begin{aligned} 0 \leq y \leq 1 \\ 1 \leq x \leq e^y \end{aligned}$$

as it was for the original $\mathrm{d}x\mathrm{d}y$ integral, but also by

$$\begin{aligned} 1 \leq x \leq e \\ \log\{x\} \leq y \leq 1 \end{aligned}$$

which is the description we need for the new $\mathrm{d}y\mathrm{d}x$ integral. We can now write that

$$\int_0^1 \int_1^{e^y} f(x, y)\mathrm{d}x\mathrm{d}y = \int_1^e \int_{\log\{x\}}^1 f(x, y)\mathrm{d}y\mathrm{d}x$$

In general, to solve these double integrals we need to remember some points:

1. After we have set up the integral, visualise the region that is encoded by the bounds in the integral. If in case, that region does not look as specified in the problem, double check the entire setup.
2. Make sure that the bounds of the outer integral are numbers or constants.
3. Make sure that the bounds of the inner integral depend only on the integration variable of the outer integral. In particular, the bounds must not depend on the integration variable of the inner integral.

Tasks

Task 5.

Calculate the volume under the surface $z = 3 + x^2 - 2y$ over the region R defined by $0 \leq x \leq 1$ and $-x \leq y \leq x$.

Task 6.

Evaluate the following integrals by first reversing the order of integration.

(a) $\int_0^3 \int_{x^2}^9 x^3 e^{y^3} \mathrm{d}y\mathrm{d}x$

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\mathrm{d}x\mathrm{d}y$

Self help

[Double integrals over a non-rectangular region (Khan Academy material)

https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-over-non-rectangular-regions

Double integrals over a non-rectangular region (YouTube video)

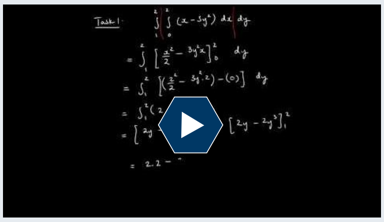
https://www.youtube.com/watch?v=k7ND70gFTLU

Changing order of integration in a double integral (YouTube video)

https://www.youtube.com/watch?v=NETmfwOAKpQ

Answers to tasks

Answer to Task 1.



Video model answers

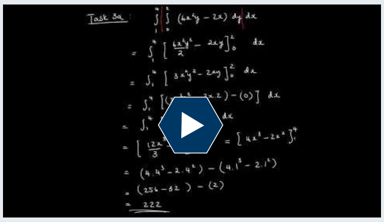
https://youtu.be/ex7OamSsKw0

Duration: 2m05s

$$\begin{aligned}\int\int_R x - 3y^2 \mathrm{d}x\mathrm{d}y &= \int_1^2\int_0^2 x - 3y^2 \mathrm{d}x\mathrm{d}y \\ &= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^2 \mathrm{d}y \\ &= \int_1^2 2 - 6y^2 \,\mathrm{d}y \\ &= \left[2y - 2y^3\right]_{y=1}^2 \\ &= -12\end{aligned}$$

Answer to Task 2. (a) $500y^3$ and $3x^2$
(b) $5y + \frac{25}{2}e^y$ and $\frac{1}{2} + ex - x$

Answer to Task 3.



Video model answers for part (a)

https://youtu.be/IWdIRzxugAs

Duration: 2m21s

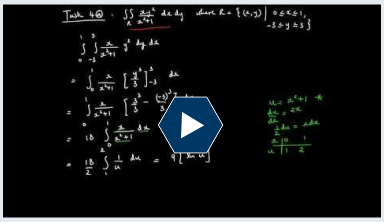
Video model answers for part (c)

https://youtu.be/Rf_5ZtJcaQI

Duration: 4m45s

(a) 222
(b) $10.5\log 2$
(c) $\frac{31}{30}$

Answer to Task 4.



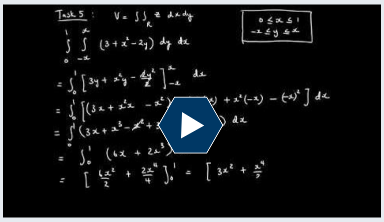
Video model answers for part (a)

https://youtu.be/5IiOmNw90XA

Duration: 2m47s

(a) $9 \log 2$
(b) $\frac{1}{2}(e^{-6} + 5)$

Answer to Task 5.



Video model answers

https://youtu.be/W-Iiz1eHNi4

Duration: 2m48s

The volume V is the double integral of $3 + x^2 - 2y$

$$\begin{aligned} V &= \int \int_R (3 + x^2 - 2y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \int_{-x}^{x} (3 + x^2 - 2y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 \left[3y + x^2 y - y^2\right]_{y=-x}^{y=x} \mathrm{d}x \\ &= \int_0^1 (6x + 2x^3)\mathrm{d}x \\ &= \left[\frac{6x^2}{2} + \frac{2x^4}{4}\right]_{x=0}^{x=1} \\ &= \frac{7}{2}. \end{aligned}$$

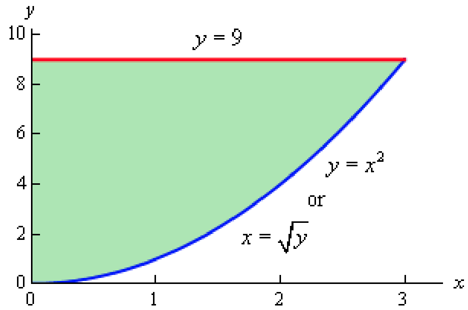
Answer to Task 6. (a)

We need to compute the integral $\int_0^3 \int_{x^2}^9 x^3 e^{y^3} \mathrm{d}y\mathrm{d}x$.

Notice that if we try to integrate with respect to y we can't calculate the integral because we can't integrate e^{y^3}. Let's hope that if we reverse the order, the integrals will be easier to solve.

Again, when we say that we're going to reverse the order of integration this means that we want to integrate with respect to x first and then y. Note as well that we can't just interchange the integrals, keeping the original limits, and be done with it. This would not fix our original problem and in order to integrate with respect to x we can't have x's in the limits of the integrals. Even if we ignored that the answer would not be a constant as it should be.

So, let's see how we reverse the order of integration. The best way to reverse the order of integration is to first sketch the region given by the original limits of integration.



From the integral we see that the inequalities that define this region are,

$$0 \le x \le 3$$
$$x^2 \le y \le 9$$

These inequalities tell us that we want the region with $y = x^2$ on the lower boundary and $y = 9$ on the upper boundary that lies between $x = 0$ and $x = 3$.

Since we want to integrate with respect to x first we will need to determine limits of x (probably in terms of y) and then get the limits on the y's. These are:

$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 9$$

Any horizontal line drawn in this region will start at $x = 0$ and end at $x = \sqrt{y}$ and so these are the limits on the x's and the range of y's for the region is 0 to 9.

Reversing the order of integration we now have that

$$\int_0^3 \int_{x^2}^9 x^3 e^{y^3} \mathrm{d}y\mathrm{d}x = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3} \mathrm{d}x\mathrm{d}y.$$

so we can now start calculating the integral on the right-hand side.

$$\begin{aligned}\int_0^9 \left[\int_0^{\sqrt{y}} x^3 e^{y^3} \mathrm{d}x\right]\mathrm{d}y &= \int_0^9 e^{y^3}\left[\frac{x^4}{4}\right]_{x=0}^{x=\sqrt{y}} \mathrm{d}y \\ &= \int_0^9 e^{y^3}\frac{y^2}{4}\mathrm{d}y\end{aligned}$$

Although this looks tricky, having y^2 multiplied with e^{y^3} makes life easier. Why?

If we substitute $u = y^3$ we get $\mathrm{d}u = 3y^2\mathrm{d}y$. The new lower limit will now be $u = 0^3 = 0$ and the new upper limit will be $u = 9^3 = 729$. We now have

$$\begin{aligned}\int_0^9 \frac{1}{4}e^{y^3}y^2\mathrm{d}y &= \int_0^{729} \frac{1}{4}e^u \frac{\cancel{y^2}}{3\cancel{y^2}}\mathrm{d}u \\ &= \int_0^{729} \frac{1}{12}e^u\mathrm{d}u \\ &= \frac{1}{12}\left(e^{729} - 1\right)\end{aligned}$$

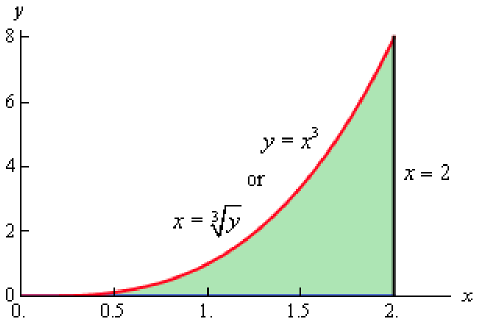
(b)

In this part we compute $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\mathrm{d}x\mathrm{d}y$.

As with the first integral we cannot calculate it by integrating with respect to x first so we'll hope that by reversing the order of integration we will get something that we can integrate. Here are the limits for the variables that we get from this integral:

$$\sqrt[3]{y} \leq x \leq 2$$
$$0 \leq y \leq 8$$

and here is a sketch for the region



If we now reverse the order of integration we will get the following region:

$$0 \leq x \leq 2$$
$$0 \leq y \leq x^3$$

The integral is then,

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,\mathrm{d}x\mathrm{d}y = \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\,\mathrm{d}y\mathrm{d}x$$

and

$$\begin{aligned}
\int_0^2 \int_0^{x^3} \sqrt{x^4+1}\,\mathrm{d}y\mathrm{d}x &= \int_0^2 \Big[\int_0^{x^3} \sqrt{x^4+1}\,\mathrm{d}y\Big]\mathrm{d}x \\
&= \int_0^2 \big[y\sqrt{x^4+1}\big]_{y=0}^{y=x^3}\,\mathrm{d}x \\
&= \int_0^2 x^3\sqrt{x^4+1}\,\mathrm{d}x
\end{aligned}$$

A tricky integral until we notice that we can use the substitution method. If we substitute $u = x^4 + 1$ we will get $\mathrm{d}u = 4x^3\mathrm{d}y$. The new lower limit will now be $u = 0^4 + 1 = 1$ and the new upper limit will be $u = 2^4 + 1 = 17$. We now have

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^4+1}\,\mathrm{d}x &= \int_0^{64} \cancel{x^3}\sqrt{u}\,\frac{1}{4\cancel{x^3}}\mathrm{d}u \\
&= \int_0^{64} \frac{1}{4}\sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{4}\Big[\frac{u^{3/2}}{\frac{3}{2}}\Big]_{u=0}^{u=64} \\
&= \frac{1}{6}\left(17^{3/2} - 1\right)
\end{aligned}$$