Preliminary Mathematics for online MSc programmes in Data Analytics

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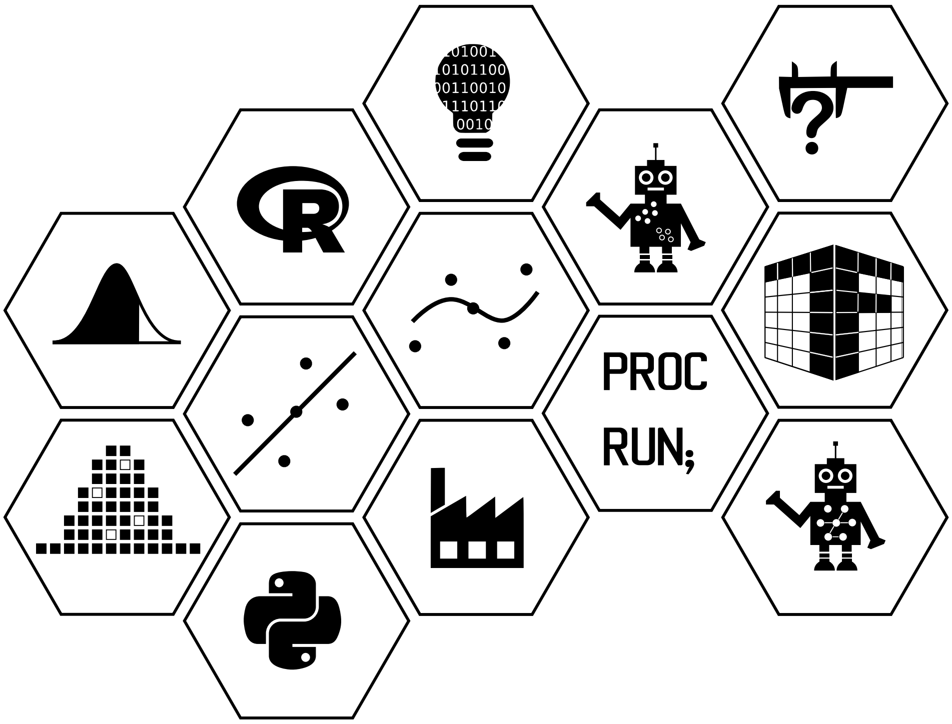
Unit 6:

Vectors and Matrices



DATA ANALYTICS GLASGOW

Vectors

Introduction to vectors

A vector is an ordered set of numbers. These could be expressed as a row

$$\begin{bmatrix} 6 & 0 & 5 & \dots & 1 \end{bmatrix},$$

or as a column

$$\begin{bmatrix} 6 \\ 0 \\ 5 \\ \vdots \\ 1 \end{bmatrix}.$$

The number of elements in a vector is referred to as its dimension. An n-dimensional vector can be represented as a row vector.

We use subscripts to denote the individual entries of a vector x_i denotes the i-th entry of the vector $\mathbf{x}$. If we had called the above vector $\mathbf{x}$ then $x[3] = 5$.

Operations on vectors As a matter of definition, when we add two vectors, we add them element by element. If

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \text{and} \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

we then have that

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Scalar multiplication is an operation that takes a number (or scalar) γ and a vector $\mathbf{x}$ and produces

$$\gamma \cdot \mathbf{x} = \begin{bmatrix} \gamma \cdot x_1 \\ \gamma \cdot x_2 \\ \gamma \cdot x_3 \\ \vdots \\ \gamma \cdot x_n \end{bmatrix}.$$

The difference $\mathbf{x} - \mathbf{y}$ can be written as $\mathbf{x} + (-1) \cdot \mathbf{y}$. Thus we need to multiply the second vector with -1 and then add the two vectors.

A **null vector** is a vector whose elements are all zero. The difference between any vector and itself yields the null vector.

A **unit vector** is a vector whose length or modulus is 1, i.e. $\sqrt{\sum_{i=1}^{n} x_i^2} = 1$.

Linear combination of vectors Given n-vectors, $\mathbf{x}$ $\mathbf{y}$ and $\mathbf{z}$, as well as scalars γ and δ, we say that $\mathbf{z}$ is a **linear combination** of $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{z} = \gamma\mathbf{x} + \delta\mathbf{y}$.

Specifically, for column vectors we have

$$\gamma \cdot \mathbf{x} + \delta \cdot \mathbf{y} = \begin{bmatrix} \gamma \cdot x_1 + \delta \cdot y_1 \\ \gamma \cdot x_2 + \delta \cdot y_2 \\ \gamma \cdot x_3 + \delta \cdot y_3 \\ \vdots \\ \gamma \cdot x_n + \delta \cdot y_n \end{bmatrix}.$$

Inner product of vectors Given two n-vectors, $\mathbf{x}$ and $\mathbf{y}$, their **inner product** (sometimes called the dot product) is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_n \cdot y_n = \sum_{i=1}^{n} x_i \cdot y_i.$$

Orthogonality of vectors Two vectors are said to be **orthogonal** if their inner product is zero.

Norm of a vector The square-root of the inner product of a vector $\mathbf{x}$ and itself is called the **norm** of a vector:

$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Linear (in)dependence of vectors

- A set of vectors is **linearly independent** if no vector in the set is
  (a) a scalar multiple of another vector in the set or
  (b) a linear combination of other vectors in the set.
- A set of vectors is **linearly dependent** if any vector in the set is
  (a) a scalar multiple of another vector in the set or
  (b) a linear combination of other vectors in the set.

Example 1 (Linear (in)dependence of vectors).

$$a = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \quad c = \begin{bmatrix} 5 & 7 & 9 \end{bmatrix}$$
$$d = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \quad e = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Note that:

- Vectors a and b are linearly independent, because neither vector is a scalar multiple of the other.
- Vectors a and d are linearly dependent, because d is a scalar multiple of a; since $d = 2a$.
- Vector c is a linear combination of vectors a and b, because $c = a + b$. Therefore, the set of vectors a, b, and c is linearly dependent.
- Vectors d, e, and f are linearly independent, since no vector in the set can be derived as a scalar multiple or a linear combination of any other vectors in the set.

Tasks

Task 1.

Find the vector $2\boldsymbol{u} - \boldsymbol{v}$ when $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ -4 \\ 7 \end{bmatrix}$.

Task 2.

Are the following vectors orthogonal?

(a) $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

Task 3.

Find the value of n such that the vectors $\boldsymbol{u} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} n \\ 1 \\ -8 \end{bmatrix}$ are orthogonal.

Self-help

Introduction to vectors: part I

https://www.mathsisfun.com/algebra/vectors.html

Introduction to vectors: part II (Khan Academy video)

https://www.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/v/vector-introduction-linear-algebra

Matrix algebra

Introduction to matrix algebra

A **matrix** is a rectangular array of numbers

$$\boldsymbol{A} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}.$$

The notational subscripts in the typical element x_{ij} refer to its row and column location in the array: specifically, x_{ij} is the element in the i-th row and the j-th column. This matrix has m rows and n columns, so is said to be of dimension $m \times n$. A matrix can be viewed as a set of column vectors, or alternatively as a set of row vectors. A vector can be viewed as a matrix with only one row or column.

Some special matrices

- A matrix with the same number of rows as columns is said to be a **square matrix**.
- Matrices that are not square are said to be **rectangular matrices**.
- A **null matrix** is composed of all 0's and can be of any dimension.
- An **identity matrix** is a square matrix with 1's on the main diagonal, and all other elements equal to 0. Formally, we have $x_{ii} = 1$ for all i and $x_{ij} = 0$ for all $i \neq j$. Identity matrices are often denoted by the symbol $\boldsymbol{I}$ (or sometimes as $\boldsymbol{I}_n$ where n denotes the dimension). The three-dimensional identity matrix isz

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- A square matrix is said to be **symmetric** if $x_{ij} = x_{ji}$.
- A **diagonal matrix** is a square matrix whose non-diagonal entries are all zero. That is $x_{ij} = 0$ for $i \neq j$.
- An **upper-triangular matrix** is a square matrix in which all entries below the diagonal are 0. That is $x_{ij} = 0$ for $i > j$.
- A **lower-triangular matrix** is a square matrix in which all entries above the diagonal are 0. That is $x_{ij} = 0$ for $i < j$.

Matrix operations

- Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if they have the same dimensions and if each element of $\boldsymbol{A}$ equals the corresponding element of $\boldsymbol{B}$.
- For any matrix $\boldsymbol{A}$, the transpose, denoted by $\boldsymbol{A}^\intercal$ (or $\boldsymbol{A}'$) is obtained by interchanging rows and columns. That is, the i-th row of the original matrix forms the i-th column of the transpose matrix. Note that if $\boldsymbol{A}$ is of dimension $m \times n$, its transpose is of dimension $n \times m$. Finally the transpose of a transpose of a matrix will yield the original matrix, i.e. $(\boldsymbol{A}^\intercal)^\intercal = \boldsymbol{A}$.
- We can add two matrices as long as they are of the same dimension. Let's assume that we have matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of dimension $m \times n$; their sum is defined as an $m \times n$ matrix $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. For instance,

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} \\ x_{21} + y_{21} & x_{22} + y_{22} \end{bmatrix}$$

- The transpose of a sum of matrices is the sum of the transpose matrices i.e. $(\boldsymbol{A} + \boldsymbol{B})^\intercal = \boldsymbol{A}^\intercal + \boldsymbol{B}^\intercal$.
- Multiplying the matrix by a scalar involves multiplying each element of the matrix by that scalar.

Example 2 (Transpose of a matrix).

The transpose of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is

$$\mathbf{A}^{\top} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{\top} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Matrix multiplication

Matrix multiplication is an operation on pairs of matrices that satisfy certain restrictions. The restriction is that the first matrix must have the same number of columns as the number of rows in the second matrix. When this condition holds the matrices are said to be **conformable** under multiplication. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ an $n \times p$ matrix. As the number of columns in the first matrix and the number of rows in the second both equal n, the matrices are conformable.

The product matrix $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$ is an $m \times p$ matrix whose ij-th element equals the inner product of the i-th row vector of the matrix $\boldsymbol{A}$ and the j-th column of matrix $\boldsymbol{B}$.

Example 3 (Multiplication of matrices).

Consider the matrices

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Suppose we want to calculate $\boldsymbol{AB}$. The matrices are conformal, as the number of rows of $\boldsymbol{A}$ (2) matches the number of columns of the matrix $\boldsymbol{B}$ (also 2).

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 & 2 \cdot 3 + 1 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 & 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 8 & 5 \end{bmatrix}.$$

Note that matrix multiplication is not commutative. The matrix product $\boldsymbol{BA}$ is also conformal (number of columns of $\boldsymbol{B}$ (3) matches the number of rows of $\boldsymbol{A}$ (also 3)), but the product $\boldsymbol{BA}$ not only contain different numbers from the product $\boldsymbol{AB}$, it even has a different dimension!

$$\boldsymbol{B} \cdot \boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \Dashv \vDash \begin{bmatrix} 1 \cdot 2 + 3 \cdot 1 & 1 \cdot 1 + 3 \cdot 2 & 1 \cdot 1 + 3 \cdot 3 \\ 2 \cdot 2 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 1 + 1 \cdot 3 \\ 1 \cdot 2 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 2 & 1 \cdot 1 + 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 10 \\ 5 & 4 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

Properties of matrix multiplication

- Even when matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable so that $\boldsymbol{A} \cdot \boldsymbol{B}$ exists, $\boldsymbol{B} \cdot \boldsymbol{A}$ may not exist.
- Even when both product matrices exist, they may not have the same dimensions.
- Even when both product matrices are of the same dimension, they may not be equal.
- If $\boldsymbol{A} \cdot \boldsymbol{B}$=$\mathbf{0}$; that does not imply either $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$ (as it would in the case of multiplying scalars).
- If $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{C}$ and $\boldsymbol{A} \neq \mathbf{0}$, that does not imply $\boldsymbol{B} = \boldsymbol{C}$.
- Matrix multiplication is associative: $(\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$.
- Matrix multiplication is distributive across sums:

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C},$$
$$(\boldsymbol{B} + \boldsymbol{C}) \cdot \boldsymbol{A} = \boldsymbol{B} \cdot \boldsymbol{A} + \boldsymbol{C} \cdot \boldsymbol{A}.$$

* Multiplication with the identity matrix yields the original matrix.

- Multiplication with the null matrix yields a null matrix.

- If $\boldsymbol{A}$ is a square matrix we can multiply the matrix by itself and get $\boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A}$. Similarly, we get $\boldsymbol{A}^n = \underbrace{\boldsymbol{A} \cdot \boldsymbol{A} \cdot \ldots \cdot \boldsymbol{A}}_{n \text{ times}}$.
- A square matrix $\boldsymbol{A}$ is said to be **idempotent** if $\boldsymbol{A} \cdot \boldsymbol{A} = \boldsymbol{A}$.
- From now on, we will refer to the product of two matrices, $\boldsymbol{A}$ and $\boldsymbol{B}$, as $\boldsymbol{AB}$ and drop the $\cdot$ notation.

Linear dependence and rank

- The **rank** of a matrix is the number of linearly independent rows (or columns) in the matrix.
- It doesn't matter whether you work with the rows or the columns.
- The rank cannot be larger than the smaller of the number of rows and columns of a matrix. In other words, for an $m \times n$ matrix, i.e. a matrix with m rows and n columns, the maximum rank of the matrix is m if $m \leq n$, or n if $n < m$.
- If the rank is equal to the smaller of the number of rows and columns the matrix is said to be of **full rank**

The rank of a matrix can be interpreted as the **dimension** of the linear space spanned by the columns (or rows) of the matrix.

Example 4.

Find the rank of the matrix by looking at both rows and columns.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

The second row is a copy of the first row and the fourth row is a scalar (-1) multiple of the third row. The first and third row however are linearly independent, hence the rank of the matrix must be 2.

We obtain the same answer by looking at the columns. The first two columns are linearly independent, but the first column is the sum of the second and the third column, hence it is not linearly independent. Thus, the rank of the matrix must be 2.

If we define the matrix $\boldsymbol{C} = \boldsymbol{AB}$ as the product of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, then

- the rank of $\boldsymbol{C}$ cannot be larger than the rank of $\boldsymbol{A}$ or the rank of $\boldsymbol{B}$.
- if $\boldsymbol{A}$ and $\boldsymbol{B}$ are of full rank then the rank of $\boldsymbol{C}$ is the smaller of the rank of $\boldsymbol{A}$ and the rank of $\boldsymbol{B}$.

Determinant

Calculating determinants of 2x2 matrices The **determinant** of a square $n \times n$ matrix $\boldsymbol{A}$ is a one-dimensional measurement of the "volume" of a matrix.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

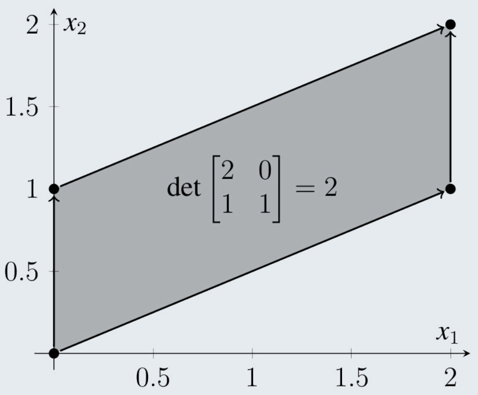
Note that there are two ways to denote the determinant of a matrix $\boldsymbol{A}$. That is $\det(\boldsymbol{A})$ or $|\boldsymbol{A}|$.

Example 5.

The determinant of the matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ is

$$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \times 1 - 0 \times 1 = 2$$

In general, one can show that the absolute value of the determinant of a matrix $\boldsymbol{A}$ is the volume of the parallelepiped spanned by its columns. In our example the (absolute value of the) determinant is the surface of the parallelogram spanned by the vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, i.e. the parallelogram with vertices in the points $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.



There is also a closed-form formula for the determinant of a 3×3 matrix. Beyond dimension three there are no simple formulae. However if the matrix $\boldsymbol{A}$ is diagonal, then the determinant is simply the product of the diagonal elements of the matrix.

Calculating determinants of 3x3 matrices The determinant of a 3×3 matrix $\boldsymbol{A}$, is defined as

$$|\boldsymbol{A}| = \begin{vmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{vmatrix} = A_{11}\cdot(A_{22}\cdot A_{33}-A_{23}\cdot A_{32}) - A_{12}\cdot(A_{21}\cdot A_{33}-A_{23}\cdot A_{31}) + A_{13}\cdot(A_{21}\cdot A_{32}-A_{22}\cdot A_{31}).$$

Supplementary material:
Higher-order determinants

These operations can be represented more conveniently using the notion of **minors**. For any square matrix $\boldsymbol{A}$, consider the sub-matrix $\boldsymbol{A}_{(ij)}$ formed by deleting the i-th row and j-th column of $\boldsymbol{A}$. The determinant of the sub-matrix $\boldsymbol{A}_{(ij)}$ is called the (i,j)-th minor of the matrix (or sometimes the minor of element A_{ij}). We denote this as M_{ij}. For instance, the minors associated with the first row of a 3×3 matrix are

$$M_{11}=\begin{vmatrix}A_{22} & A_{23}\\A_{32} & A_{33}\end{vmatrix}, M_{12}=\begin{vmatrix}A_{21} & A_{23}\\A_{31} & A_{33}\end{vmatrix}, M_{13}=\begin{vmatrix}A_{21} & A_{22}\\A_{31} & A_{32}\end{vmatrix}.$$

Recalling how we specified determinants of order 2 and 3, we see that

$$\det \boldsymbol{A} = A_{11}\cdot M_{11} - A_{12}\cdot M_{12} + A_{13}\cdot M_{13}.$$

Note the alternating positive and negative signs.

Supplementary material:
Cofactor matrix

For any element A_{ij} of a square matrix $\boldsymbol{A}$, the cofactor element is given by

$$C_{ij} = (-1)^{i+j}|M_{ij}|$$

where M_{ij} is the (i,j)-th minor of the matrix (and $|M_{ij}|$ refers to the determinant of that matrix). Thus, if we want to calculate C_{12} we have that

$$\begin{aligned}C_{12} &= (-1)^{1+2}|M_{12}|\\&= (-1)^{3}|M_{12}|\\&= -|M_{12}|\end{aligned}$$

The **cofactor** matrix $\boldsymbol{C}$ is obtained by replacing each element of the matrix $\boldsymbol{A}$ by its corresponding cofactor element C_{ij}.

Let's assume that

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}.$$

The cofactors are

$$\begin{aligned} C_{11} &= (-1)^{1+1}|M_{11}| = -1 \\ C_{12} &= (-1)^{1+2}|M_{12}| = -4 \\ C_{21} &= (-1)^{2+1}|M_{21}| = -2 \\ C_{22} &= (-1)^{2+2}|M_{22}| = 3 \end{aligned}$$

since in this case $M_{11} = -1, M_{12} = 4, M_{21} = 2, M_{22} = 3$. Thus, we have that

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -2 & 3 \end{bmatrix}.$$

Properties of determinants

- One can show that the determinant of a product of square matrices is the product of the determinants

  $$\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}).$$

  This implies that we can exchange the order of multiplication inside the determinant, as long as we end up with conformant matrix multiplications, so for example

  $$\det(\boldsymbol{ABC}) = \det(\boldsymbol{A})\det(\boldsymbol{B})\det(\boldsymbol{C}) = \det(\boldsymbol{B})\det(\boldsymbol{A})\det(\boldsymbol{C}) = \det(\boldsymbol{BAC}).$$

- The determinant of $\boldsymbol{A}^\intercal$ is the same as the determinant of $\boldsymbol{A}$, i.e. $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^\intercal)$.
- The determinant of a matrix is non-zero if and only if the matrix is of full rank.

Trace of a matrix

The **trace** of a square matrix is the sum of its diagonal elements, i.e. the trace of the $n \times n$ matrix $\boldsymbol{A}$ is

$$\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} A_{ii}.$$

Example 6.

The *trace* of the matrix $\begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$ is $4 + 1 = 5$.

One can show that for conformable matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, $\operatorname{tr}(\boldsymbol{ABC}) = \operatorname{tr}(\boldsymbol{CAB}) = \operatorname{tr}(\boldsymbol{BCA})$, as long as we end up with conformant matrix multiplications. Note however that in general, $\operatorname{tr}(\boldsymbol{ABC}) \neq \operatorname{tr}(\boldsymbol{ACB})$ and $\operatorname{tr}(\boldsymbol{ABC}) \neq \operatorname{tr}(\boldsymbol{BAC})$. In other words, when computing the trace of a product of matrices we can move the first matrix to the end and vice versa, but cannot swap the order of terms as freely as we can for the determinant.

The inverse of a matrix

Definition For a square matrix $\boldsymbol{A}$, there may exist a matrix $\boldsymbol{B}$ such that

$$\boldsymbol{AB} = \boldsymbol{BA} = I.$$

An **inverse**, if it exists is denoted as $\boldsymbol{A}^{-1}$, so the above definition can be written as

$$\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = I.$$

If an inverse does not exist for a matrix, the matrix is said to be **singular**. If an inverse exists, the matrix is said to be **non-singular**. One can show that square matrices are non-singular if and only if they are of full rank.

Properties of inverse matrices

- The inverse of an inverse yields the original matrix: $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$.
- The inverse of a product is the product of inverses with order switched: $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
- The inverse of a transpose is the transpose of the inverse: $\left(\boldsymbol{A}^{\top}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\top}$.

Calculating inverses of 2x2 matrices Calculating inverses of matrices can be time-consuming, but a simple formula exists for 2×2 matrices. If $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\boldsymbol{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

Note that the denominator in the multiplicative constant is just the determinant of $\mathbf{A}$.

Example 7.

$$\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}^{-1} = \underbrace{\frac{1}{3 \times 7 - (-5) \times (-4)}}_{=1}\begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$$

Calculating inverses of diagonal matrices If $\boldsymbol{A}$ is a diagonal matrix, then $\boldsymbol{A}^{-1}$ is also diagonal, with diagonal elements $\frac{1}{A_{ij}}$.

Supplementary material:
Inverses of matrices of matrices of arbitrary dimension

For any square matrix $\boldsymbol{A}$, the **adjoint** of $\boldsymbol{A}$ is given by the transpose of the cofactor matrix. Denoting the associated cofactor matrix as $\boldsymbol{C}$, we have adj$(\boldsymbol{A}) = \boldsymbol{C}^{\top}$.

For any square matrix $\boldsymbol{A}$, the inverse $\boldsymbol{A}^{-1}$ is given by

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\text{adj}(\boldsymbol{A}),$$

which is defined as long as $\det(\boldsymbol{A}) \neq 0$.

An alternative approach would be to use Gaussian elimination.

Orthogonal matrices

A matrix $\boldsymbol{A}$ which satisfies the condition that $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\top}$ is said to be **orthogonal**.

In an orthogonal matrix the columns (or rows) are orthogonal (i.e. their inner product is 0) and the columns (or rows) have unit length (i.e. the squares of their entries sum to 1).

Quadratic forms

Quadratic forms play an important role in determining key properties of matrices.

Consider a quadratic function on $\mathbb{R}^2$: $f(x_1, x_2) = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$.

(a_{12} and a_{21} are not uniquely defined as we only use their sum, so we can choose $a_{12} = a_{21}$, making the resulting matrix symmetric.)

Then we can define

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \text{ and} \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and rewrite $f(x_1, x_2)$ in terms of matrix-vector products.

This can be expressed in a matrix representation as

$$\begin{aligned} f(x_1, x_2) &= \mathbf{x}^\top \boldsymbol{A}\mathbf{x} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a21x_1 + a_{22}x_2 \end{bmatrix} \\ &= x_1 \times (a_{11}x_1 + a_{12}x_2) + x_2 \times (a21x_1 + a_{22}x_2) \\ &= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \end{aligned}$$

Sign definiteness of quadratic forms Given an $n \times n$ square matrix $\boldsymbol{A}$ and a quadratic form $\mathbf{x}^\top \boldsymbol{A}\mathbf{x}$; the matrix is said to be:

- positive definite if $\mathbf{x}^\top \boldsymbol{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
- positive semi-definite if $\mathbf{x}^\top \boldsymbol{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
- negative definite if $\mathbf{x}^\top \boldsymbol{A}\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
- negative semi-definite if $\mathbf{x}^\top \boldsymbol{A}\mathbf{x} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
- indefinite if $\mathbf{x}^\top \boldsymbol{A}\mathbf{x} > 0$ for some $\mathbf{x}$ in $\mathbf{R}^n$ and $\mathbf{x}^\top \boldsymbol{A}\mathbf{x} < 0$ for some others $\mathbf{x}$ in $\mathbb{R}^n$.

Eigenvectors and eigenvalues

Consider a square matrix $\boldsymbol{A}$. The non-zero vector $\mathbf{v}$ is called an **eigenvector** of $\boldsymbol{A}$ corresponding to the (scalar) **eigenvalue** λ if

$$\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}$$

Note that if $\mathbf{v}$ satisfies the above definition then any scalar multiple of $\mathbf{v}$ will satisfy the above as well. Thus we will introduce the convention that we choose $\mathbf{v}$ such that is has unit length, $\|\mathbf{v}\|^2 = \sum_{i=1}^n v_i^2 = 1$.

To find eigenvalues and eigenvectors we rewrite the previous definition as

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\mathbf{v} = \mathbf{0}$$

where $\boldsymbol{I}$ is the identity matrix. A homogeneous system of linear equations only has non-zero solutions if the matrix defining the system of linear equations is not of full rank, i.e.

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

This determinant is a polynomial ("characteristic polynomial"), which can be solved for λ, yielding the eigenvalues. These can the plugged into the first definition in order to obtain the eigenvectors.

Note that in general, eigenvalue and eigenvectors can contain complex numbers. However, if the matrix $\boldsymbol{A}$ is symmetric, this won't be the case.

Example 8 (Finding eigenvalues and eigenvectors).

Let's find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$.

We first have to find

$$\det\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - \lambda\boldsymbol{I}\right) = \det\begin{bmatrix} -\lambda & 4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 4 = 0.$$

This yields $\lambda_1 = 2$ and $\lambda_2 = -2$.

To find the eigenvector corresponding to $\lambda_1 = 2$ we need to solve the linear system

$$\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - 2 \cdot \boldsymbol{I}\right)\mathbf{v} = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

thus the corresponding eigenvector is proportional to $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Why?

(*Hint*: Try replacing $\mathbf{v}$ with $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and solve the previous equation)

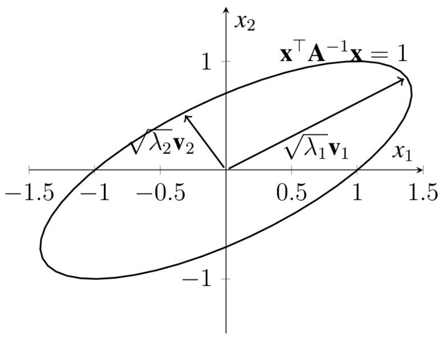
To find the eigenvector corresponding to $\lambda_2 = -2$ we need to solve the linear system

$$\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - (-2) \cdot \boldsymbol{I}\right) \mathbf{v} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

thus the corresponding eigenvector is proportional to $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. (*Hint*: Look at the previous hint)

In practice it is much easier (and numerically more efficient and stable) to use software like R or Matlab to find eigenvectors and eigenvalues. Note that in general, eigenvectors and eigenvalues are not necessarily real-valued: they might contain complex numbers.

Interpretation of eigenvectors and eigenvalues We can interpret eigenvectors and eigenvalues as the direction and reciprocal length of the principal axis of the ellipsoid defined by $\mathbf{x}^\top \boldsymbol{A}^{-1}\mathbf{x} = 1$. The principal axes of the ellipsoid defined by the equation $\boldsymbol{x}^\top \boldsymbol{A}^{-1}\boldsymbol{x} = 1$ are given by $\sqrt{\lambda_1}\mathbf{v}_1$ and $\sqrt{\lambda_2}\mathbf{v}_2$. The figure below shows this for $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.



This has an important interpretation in Statistics. If we consider normally-distributed data with mean $\boldsymbol{\mu}$, (co-)variance matrix $\boldsymbol{\Sigma}$, then its isocontours (lines of constant density) are given by the equation $(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$. Thus the eigenvectors of the (co-)variance matrix $\boldsymbol{\Sigma}$ give the direction of largest and smallest spread of the distribution. The variances in these directions are given by the eigenvalues, or, equivalently, their standard deviations by the square root of the eigenvalues.

Relationship to other matrix properties If the matrix $\boldsymbol{A}$ is symmetric (as it often is in Statistics and Machine Learning), then eigenvalues relate to many key properties of matrices

- $\boldsymbol{A}$ is of full rank if and only if all eigenvalues are non-zero.
- $\boldsymbol{A}$ is orthogonal if its eigenvalues are -1, 0, or 1.
- $\boldsymbol{A}$ is positive (semi-)definite if all its eigenvalues are positive (non-negative).
- $\boldsymbol{A}$ is negative (semi-)definite if all its eigenvalues are negative (non-positive).
- The trace of a $\boldsymbol{A}$ is the sum of its eigenvalues and the determinant is the product of its eigenvalues.
- $\boldsymbol{A}$ and its inverse $\boldsymbol{A}^{-1}$ have the same eigenvectors, but the eigenvalues of the inverse are the reciprocals of the eigenvalues of the matrix $\mathbf{A}$.

Supplementary material:
Finding square roots of matrices

Let A be a square matrix. A matrix B, where $B^2 = A$, is called a **square root** of A.

Let us consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

and matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since $B^2 = A$, we have:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

This gives us:

$$\begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & cb + d^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

We now end up with four equations and four unknowns to find, and this is, if not difficult, certainly a rather tedious process. We note that the square root of a **diagonal** matrix can be found more easily. Recall that a diagonal matrix is a square matrix in which the non-diagonal entries are all zero.

The matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ has the following square roots:

$$\begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{bmatrix}, \begin{bmatrix} -\sqrt{a} & 0 \\ 0 & \sqrt{b} \end{bmatrix}, \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{b} \end{bmatrix}, \begin{bmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{b} \end{bmatrix}$$

As the matrix A is not diagonal to start with, we can utilise a technique called **diagonalisation** to help us.

The first step towards this is to find the eigenvalues and eigenvectors of A.

We note that:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

This simplifies to:

$$(2-\lambda)(2-\lambda) - 4 = \lambda(\lambda - 4) = 0$$

Thus, the eigenvalues of A are **0** and **4**.

Since A has two distinct eigenvalues, it is **diagonalisable**.

The next step is to find eigenvectors by solving $A\mathbf{x} = 0$.

$$\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

By substituting in the eigenvalue $\lambda = 0$, we find the corresponding eigenvector is $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Similarly substituting in the eigenvalue $\lambda = 4$, we find the corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If A is diagonalisable, then by definition, there must be an invertible matrix S, such that $D = S^{-1}AS$ is diagonal. We also know (from a theorem) that the column vectors of S must coincide with the eigenvectors of A.

With this in mind, we now define a non-singular matrix $S = [\mathbf{u}\ \mathbf{v}]$.

So,

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Evaluating $S^{-1}AS$ gives us the diagonalised matrix D.

$$D = S^{-1}AS = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

We now have a matrix D which is diagonal and note that the entries in the main diagonal are the same as the eigenvalues.

The square roots of D are: $D^{\frac{1}{2}} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$.

But what about the square roots of A?

Since

$$D = S^{-1}AS, \text{ we have } A = SDS^{-1}$$

Now consider:

$$(SD^{\frac{1}{2}}S^{-1})(SD^{\frac{1}{2}}S^{-1}) = SD^{\frac{1}{2}}(S^{-1}S)D^{\frac{1}{2}}S^{-1} = SD^{\frac{1}{2}}D^{\frac{1}{2}}S^{-1} = SDS^{-1} = A$$

We can see that

$$A = (SD^{\frac{1}{2}}S^{-1})^2$$

Therefore,

$$A^{\frac{1}{2}} = SD^{\frac{1}{2}}S^{-1}$$

We can now work out the square roots of A. We have:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

To summarise, the square roots of matrix A are $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$.

Note that in this example, we had 2 distinct eigenvalues (i.e., the quadratic equation had two real roots). It is possible to have repeated or complex roots, but that is outside of the scope of this course.

Tasks

Task 4.

Which of the following is not conformable under multiplication?

(a) $\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 0 & 4 \\ 0 & 2 & 5 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 & 0 \\ 5 & 0 & 2 \end{bmatrix}$

Task 5.

Evaluate $\begin{vmatrix} a+b & c+d \\ -b & -d \end{vmatrix}$

Task 6.

The multiplication of matrices $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ gives a matrix of what dimension?

Task 7.

The addition of the matrices $\begin{bmatrix} -3 & 0 & 4 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} -3 & 0 \\ 1 & 2 \end{bmatrix}$ gives a matrix of what dimension?

Task 8.

Can you find the trace and the determinant of the following matrices?

(a) $\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

Task 9.

Let $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 0 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 3 \\ 1 & 6 \\ -2 & 2 \end{bmatrix}$. Find $A + B - 2C$.

Task 10.

Let $\boldsymbol{D} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$, $\boldsymbol{E} = \begin{bmatrix} 4 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix}$, and $\boldsymbol{F} = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 3 & -2 \end{bmatrix}$

State which of the products $\boldsymbol{D}^2$, $\boldsymbol{ED}$, $\boldsymbol{DE}$, $\boldsymbol{E}^2$, $\boldsymbol{EF}$, $\boldsymbol{FE}$ and $\boldsymbol{F}^2$ exist, and work out those that do exist.

Task 11.

State whether or not the matrix product can be calculated for each pair of matrices. If so, calculate it and state the order of the resulting matrix. If not, state why the product cannot be calculated.

(a) Product $\boldsymbol{AB}$ when $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 4 & -2 \end{bmatrix}$

(b) Product $\boldsymbol{CD}$ when $\boldsymbol{C} = \begin{bmatrix} 3 & 5 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

(c) Product $\boldsymbol{EF}$ when $\boldsymbol{E} = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$ and $\boldsymbol{F} = \begin{bmatrix} 2 & 4 & 5 \\ 6 & 1 & -1 \end{bmatrix}$

(d) Product $\boldsymbol{GH}$ when $\boldsymbol{G} = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and $\boldsymbol{H} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$

Task 12.

Determine the rank of the following matrix, $\boldsymbol{D} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

Task 13.

Given the following matrix $\boldsymbol{B}$, calculate $\mathrm{tr}(\boldsymbol{B})$:

$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 & 4 & 2 \\ -4 & 11 & 5 & 2 & 0 \\ -1 & 0 & 3 & \pi & 3 \\ 22 & 5 & 3 & 1 & 1 \\ 3 & 5 & -22 & 1 & 14 \end{bmatrix}$$

Task 14.

Find the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Task 15.

Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 matrices and let $\boldsymbol{AB} = \boldsymbol{BA} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Which of the following statements is true?

(a) $\boldsymbol{A} = \boldsymbol{B}^{-1}$
(b) $\boldsymbol{B} = \boldsymbol{A}^{-1}$
(c) $\boldsymbol{A}^{-1} = \frac{1}{3}\boldsymbol{B}$
(d) $\boldsymbol{A}^{-1} = 3\boldsymbol{B}$

Task 16.

Which of these matrices are orthogonal?

$$\boldsymbol{P} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{Q} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Task 17.

Calculate the determinant of the following matrices.

(a) $\boldsymbol{X} = \begin{bmatrix} 3 & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$

(b) $\boldsymbol{Y} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$

(c) $\boldsymbol{Z} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{4} \end{bmatrix}$

(d) $\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -2 \end{bmatrix}$

(e) $\boldsymbol{V} = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & -1 \\ 3 & 0 & 7 \end{bmatrix}$

Task 18.

Find the eigenvalues and eigenvectors of the following matrices.

(a) $\boldsymbol{X} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

(b) $\boldsymbol{Y} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

(c) $\boldsymbol{Z} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Self-help

Introduction to matrices (Khan Academy material)

https://www.khanacademy.org/math/algebra-home/alg-matrices

A minimal introduction to Matrix Algebra

http://www.maths.lth.se/matstat/kurser/masm22/Matrix%20Intro.pdf

Introductory notes on Matrix Algebra

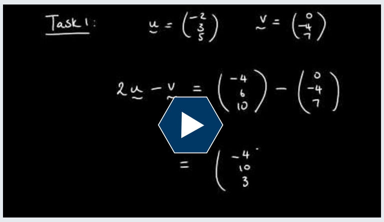
http://www2.econ.iastate.edu/classes/econ500/hallam/documents/Intro_Matrix_Algebra.pdf

Introduction to eigenvalues and eigenvectors (YouTube video)

https://www.youtube.com/watch?v=G4N8vJpf7hM

Answers to tasks

Answer to Task 1.



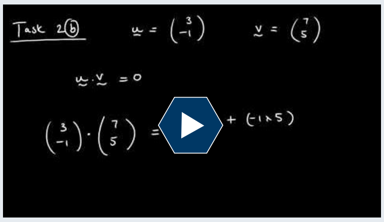
Video model answers

https://youtu.be/YyXIRjF1U8U

Duration: 0m37s

$$2\boldsymbol{u} - \boldsymbol{v} = \begin{bmatrix} 2 \times (-2) - 4 \\ 2 \times 3 - (-4) \\ 2 \times 5 - 7 \end{bmatrix} \begin{bmatrix} -4 \\ 10 \\ 3 \end{bmatrix}$$

Answer to Task 2.



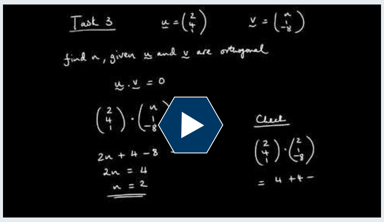
Video model answers for part (b)

https://youtu.be/XZjF39DJ-b0

Duration: 0m51s

(a) $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 \times v_1 + u_2 \times v_2 = 1 \times 2 + 2 \times (-1) = 0$, therefore orthogonal.
(b) $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 \times v_1 + u_2 \times v_2 = 3 \times 7 + (-1) \times 5 = 16 \neq 0$, therefore not orthogonal.

Answer to Task 3.



Video model answers for part (b)

https://youtu.be/p3np3_dxNXU

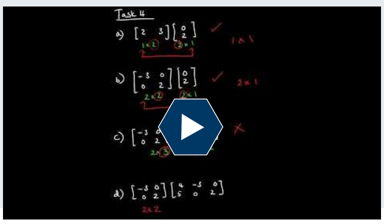
Duration: 1m06s

We want

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 \times v_1 + u_2 \times v_2 + u_3 \times v_3 = 2 \times n + 4 \times 1 + 1 \times -8 = 2n - 4 = 0$$

Hence we have to set $n = \frac{4}{2} = 2$.

Answer to Task 4.



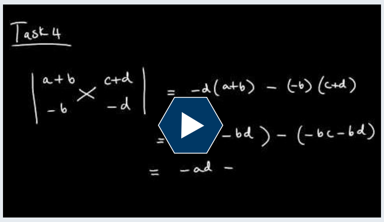
Video model answers

https://youtu.be/Y4Nevbz6XiI

Duration: 1m52s

Only in part (c) is the number of columns of the first matrix (3 columns) different from the number of rows of the second matrix (2 rows).

Answer to Task 5.



Video model answers

https://youtu.be/WY0qiOhm1RY

Duration: 0m52s

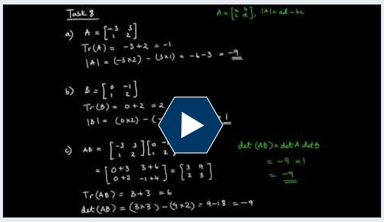
$$\begin{vmatrix} a+b & c+d \\ -b & -d \end{vmatrix} = (a+b)(-d) - (-b)(c+d) = -ad - bd + bc + bd = -ad + bc$$

Answer to Task 6. The first matrix has 3 rows and 3 columns ("3×3") and the second matrix 3 rows and 1 column ("3×1").

The matrices are conformable as the number of columns of the first matrix (3) matches the number of rows of the second matrix (also 0). The resulting matrix has the same number of rows of the first matrix and the same number of columns as the second matrix, hence it is a 3×1 matrix.

Answer to Task 7. Only matrices of the same dimension can be added, hence we cannot add these matrices together.

Answer to Task 8.



Video model answers

https://youtu.be/NA4yeR0_jlI

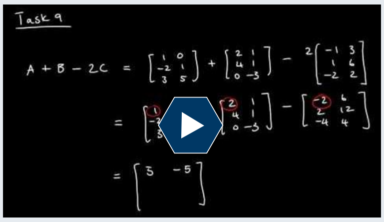
Duration: 3m06s

For part (c) we first need to multiply the two matrices together

$$\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 \cdot 0 + 3 \cdot 1 & -3 \cdot (-1) + 3 \cdot 2 \\ 1 \cdot 0 + 2 \cdot 1 & 1 \cdot (-1) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 2 & 3 \end{bmatrix}$$

$$\operatorname{tr}\left(\begin{bmatrix} -3 & 3 \\ 1 & 2 \end{bmatrix}\right) = -3 + 2 = -1 \qquad \begin{vmatrix} -3 & 3 \\ 1 & 2 \end{vmatrix} = -3 \times 2 - 1 \times 3 = -9$$
$$\operatorname{tr}\left(\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\right) = 0 + 2 = 2 \qquad \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 0 \times 2 - 1 \times (-1) = 1$$
$$\operatorname{tr}\left(\begin{bmatrix} 3 & 9 \\ 2 & 3 \end{bmatrix}\right) = 3 + 3 = 6 \qquad \begin{vmatrix} 3 & 9 \\ 2 & 3 \end{vmatrix} = 3 \times 3 - 2 \times 9 = -9$$

Answer to Task 9.



Video model answers

https://youtu.be/Q2wPJ3Hg5aA

Duration: 1m35s

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 0 & -3 \end{bmatrix} - 2\begin{bmatrix} -1 & 3 \\ 1 & 6 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 0 & -10 \\ 7 & -2 \end{bmatrix}$$

Answer to Task 10. The dimensions of the matrices are

Matrix	Dimension
$\boldsymbol{D}$	2×2
$\boldsymbol{E}$	2×3
$\boldsymbol{F}$	3×2

To calculate $\boldsymbol{D}^2$, we multiply a 2×2 matrix by a 2×2 matrix. The number of columns of the first matrix (2) matches the number of rows of the second matrix (2). Hence, the matrices are conformable and the resulting matrix is of dimension 2×2.

$$\boldsymbol{D}^2 = \boldsymbol{D} \cdot \boldsymbol{D} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + (-2) \times 3 & 1 \times (-2) + (-2) \times (-1) \\ 3 \times 1 + (-1) \times 3 & 3 \times (-2) + (-1) \times (-1) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

To calculate $\boldsymbol{ED}$, we multiply a 2×3 matrix by a 2×2 matrix. The number of columns of the first matrix (3) does not match the number of rows of the second matrix (2). Hence, the matrices are not conformable and cannot be multiplied.

To calculate $\boldsymbol{DE}$, we multiply a 2×2 matrix by a 2×3 matrix. The number of columns of the first matrix (2) matches the number of rows of the second matrix (2). Hence, the matrices are conformable and the resulting matrix is of dimension 2×3.

$$\boldsymbol{D} \cdot \boldsymbol{E} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + (-2) \times 3 & 1 \times (-1) + (-2) \times (-2) & 1 \times 2 + (-2) \times 2 \\ 3 \times 4 + (-1) \times 3 & 3 \times (-1) + (-1) \times (-2) & 3 \times 2 + (-1) \times 2 \end{bmatrix} = \begin{bmatrix} -2 & 3 & -2 \\ 9 & -1 & 4 \end{bmatrix}$$

To calculate $\boldsymbol{E}^2$, we multiply a 2×3 matrix by a 2×3 matrix. The number of columns of the first matrix (3) does not match the number of rows of the second matrix (2). Hence, the matrices are not conformable and cannot be multiplied.

To calculate $\boldsymbol{EF}$, we multiply a 2×3 matrix by a 3×2 matrix. The number of columns of the first matrix (3) matches the number of rows of the second matrix (3). Hence, the matrices are conformable and the resulting matrix is of dimension 2×2.

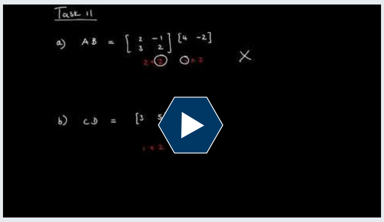
$$\boldsymbol{E} \cdot \boldsymbol{F} = \begin{bmatrix} 4 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 \times 1 + (-1) \times (-2) + 2 \times 3 & 4 \times 5 + (-1) \times 3 + 2 \times (-2) \\ 3 \times 1 + (-2) \times (-2) + 2 \times 3 & 3 \times 5 + (-2) \times 3 + 2 \times (-2) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 13 & 5 \end{bmatrix}$$

To calculate $\boldsymbol{FE}$, we multiply a 3×2 matrix by a 2×3 matrix. The number of columns of the first matrix (2) matches the number of rows of the second matrix (2). Hence, the matrices are conformable and the resulting matrix is of dimension 3×3.

$$\boldsymbol{F} \cdot \boldsymbol{E} = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 5 \times 3 & 1 \times (-1) + 5 \times (-2) & 1 \times 2 + 5 \times 2 \\ (-2) \times 4 + 3 \times 3 & (-2) \times (-1) + 3 \times (-2) & (-2) \times 2 + 3 \times 2 \\ 3 \times 4 + (-2) \times 3 & 3 \times (-1) + (-2) \times (-2) & 3 \times 2 + (-2) \times 2 \end{bmatrix} = \begin{bmatrix} 19 & -11 & 12 \\ 1 & -4 & 2 \\ 6 & 1 & 2 \end{bmatrix}$$

To calculate F^2, we multiply a 3×2 matrix by a 3×2 matrix. The number of columns of the first matrix (2) does not match the number of rows of the second matrix (3). Hence, the matrices are not conformable and cannot be multiplied.

Answer to Task 11.



Video model answers to parts (a) and (b)

https://youtu.be/ez2yaH-5je0

Duration: 2m31s

(a) Not conformable.

(b)

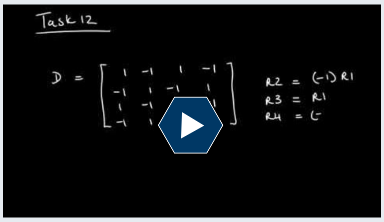
$$C \cdot D = \begin{bmatrix} 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 \times 3 + 5 \times 2 & 3 \times 2 + 5 \times 2 \end{bmatrix} = \begin{bmatrix} 19 & 16 \end{bmatrix}$$

(c)

$$E \cdot E = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 5 \\ 6 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 \times 2 + 3 \times 6 & 4 \times 4 + 3 \times 1 & 4 \times 5 + 3 \times (-1) \\ (-1) \times 2 + 2 \times 6 & (-1) \times 4 + 2 \times 1 & (-1) \times 5 + 2 \times (-1) \end{bmatrix} = \begin{bmatrix} 26 & 19 & 17 \\ 10 & -2 & -7 \end{bmatrix}$$

(d) Not conformable

Answer to Task 12.



Video model answers

https://youtu.be/B0OH4vdK6Jc

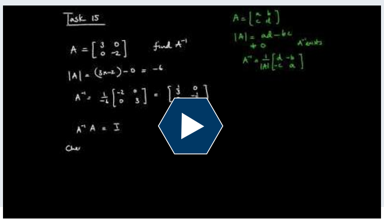
Duration: 0m44s

The second row is the negative of the first row, the third row is identical to the first row and the fourth row is identical to the second row. Hence the rank is 1.

Answer to Task 13.

$$\operatorname{tr}(B) = \operatorname{tr}\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 2 \\ -4 & 11 & 5 & 2 & 0 \\ -1 & 0 & 3 & \pi & 3 \\ 22 & 5 & 3 & 1 & 1 \\ 3 & 5 & -22 & 1 & 14 \end{bmatrix}\right) = 1 + 11 + 3 + 1 + 14 = 30$$

Answer to Task 14.



Video model answers

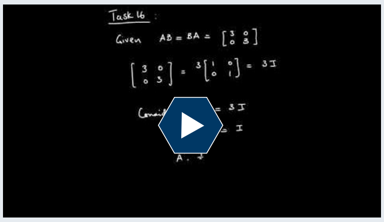
https://youtu.be/yl2H0J_UHYc

Duration: 2m35s

$$A^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}^{-1} = \frac{1}{3 \times (-2) - 0 \times 0} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

We could have also used the fact that the matrix is diagonal and simply taken the reciprocals of the diagonal elements.

Answer to Task 15.



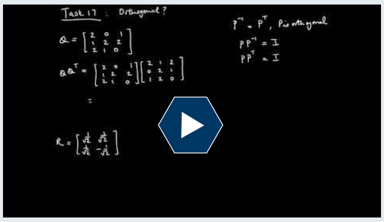
Video model answers

https://youtu.be/oK9AJa01lIE

Duration: 1m12s

We have that $\boldsymbol{AB} = 3\boldsymbol{I}$, which is the same as $\boldsymbol{A}\frac{1}{3}\boldsymbol{B} = \boldsymbol{I}$, hence $\boldsymbol{A}^{-1} = \frac{1}{3}\boldsymbol{B}$.

Answer to Task 16.



Video model answers

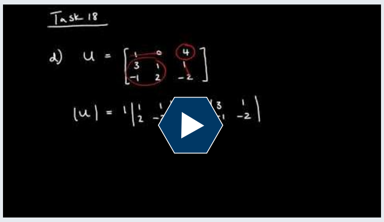
https://youtu.be/1M8BOZquQNU

Duration: 2m59s

We need to multiply the matrices by their transpose and check whether the result is the identity matrix.

This is the case for $\boldsymbol{P}$ and $\boldsymbol{R}$, hence they are orthogonal. $\boldsymbol{Q}$ is not orthogonal.

Answer to Task 17.



Video model answers for part (d)

https://youtu.be/lTkzvuSmHYQ

Duration: 1m59s

(a) $\det(\boldsymbol{X}) = \begin{vmatrix} 3 & 1 \\ 2 & \frac{1}{2} \end{vmatrix} = 3 \times \frac{1}{2} - 2 \times 1 = \frac{3}{2} - \frac{4}{2} = -\frac{1}{2}$

(b) $\det(\boldsymbol{Y}) = \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} = 4 \times 1 - 1 \times 0 = 4$

(c) $\det(\boldsymbol{Z}) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{4} \end{vmatrix} = \frac{1}{2} \times \frac{1}{4} - (-\frac{1}{3} \times \frac{1}{2}) = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}$

(d)

$$\begin{vmatrix} 1 & 0 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -2 \end{vmatrix} = 1 \times 1 \times (-2) + 0 \times 1 \times (-1) + 4 \times 3 \times 2$$
$$- (-1) \times 1 \times 4 - 2 \times 1 \times 1 - (-2) \times 3 \times 0$$
$$= -2 + 0 + 24 - (-4) - 2 - 0$$
$$= 24$$

(e)

$$\begin{vmatrix} 4 & 1 & 3 \\ 0 & 2 & -1 \\ 3 & 0 & 7 \end{vmatrix} = 4 \times 2 \times 7 + 1 \times (-1) \times 3 + 3 \times 0 \times 0$$
$$- 3 \times 2 \times 3 - 0 \times (-1) \times 4 - 7 \times 0 \times 1$$
$$= 56 + (-3) + 0 - 18 - 0 - 0$$
$$= 35$$

Do also watch the video model answers to part (d). They show how to calculate the determinant using minors, rather than the formula for 3×3 matrices given in the notes.

Answer to Task 18. (a) To find the eigenvalues of X, we first solve:

$$\det(\boldsymbol{X} - \lambda \boldsymbol{I}) = 0$$

Recall that $\lambda \boldsymbol{I} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

We have:

$$\boldsymbol{X} - \lambda \boldsymbol{I} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}$$

So continuing, we get:

$$\det(\boldsymbol{X} - \lambda \boldsymbol{I}) = (3-\lambda)(4-\lambda) - (2 \times 1) = 0$$

This simplifies to:

$$12 - 3\lambda - 4\lambda + \lambda^2 - 2 = 0$$
$$\lambda^2 - 7\lambda + 10 = 0$$
$$(\lambda - 2)(\lambda - 5) = 0$$
$$\lambda = 2, 5$$

Thus, the eigenvalues of $\boldsymbol{X}$ are **2** and **5**.

To find **eigenvectors**, we solve:

$$(\boldsymbol{X} - \lambda \boldsymbol{I})\boldsymbol{v} = 0,$$

where **v** is the eigenvector and λ is the corresponding eigenvalue. We have:

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

We substitute the eigenvalue $\lambda = 2$ into the matrix above to get:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

What we are doing is that we are solving this for $\boldsymbol{v}$, and note that this is in fact a set of equations in the components of $\boldsymbol{v}$. So, multiplying the matrices gives us:

$$v_1 + v_2 = 0$$
$$2v_1 + 2v_2 = 0$$

These equations are essentially the same and we find they both simplify out to:

$$v_1 = -v_2$$

We do not expect to find a unique solution here as we only have one equation with two unknowns. Any scalar multiple of the solution will be a another solution.

So, suppose we let $v_2 = k$, then $v_1 = -k$. Thus, for any value of k, we have:

$$\begin{bmatrix} -k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We go for the 'simplest form' of the eigenvector (the simplest non-zero multiple of this) to get:

eigenvalue $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to the eigenvector $\lambda = 2$.

Similarly, we now substitute the second eigenvalue $\lambda = 5$ into

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

to get:

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Now multiplying the matrices gives us:

$$-2v_1 + v_2 = 0$$

$$2v_1 - v_2 = 0$$

Both equations simplify out to:

$$2v_1 = v_2$$

Suppose we let $v_1 = k$, then $v_2 = 2k$. Thus, for any value of k, we have:

$$\begin{bmatrix} k \\ 2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We go for the 'simplest form' of the eigenvector (the simplest non-zero multiple of this) to get:

eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5$.

(b) $\lambda = 3$ (multiplicity of 2), $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) To find the eigenvalues of $\boldsymbol{Z}$, we first solve:

$$\det(\boldsymbol{Z} - \lambda \boldsymbol{I}) = 0$$

Recall that $\lambda \boldsymbol{I} = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

We have:

$$\boldsymbol{Z} - \lambda \boldsymbol{I} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

So continuing, we get:

$$\det(\boldsymbol{Z} - \lambda \boldsymbol{I}) = (2-\lambda)(2-\lambda)(2-\lambda) - (1 \times 0) + 0 = 0$$

This simplifies to:

$$(\lambda - 2)^3 = 0$$

$$\lambda = 2$$

Thus, the eigenvalue of $\boldsymbol{Z}$ is **2**.

(*Note that we have repeated roots here and multiplicity of 3.*)

To find the **eigenvector**, we solve $(\boldsymbol{Z} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$, where $\boldsymbol{v}$ is the eigenvector and λ is the corresponding eigenvalue. We have:

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

and substituting the eigenvalue $\lambda = 2$ into the matrix above to get:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

We are solving this for **v**, and note that this is in fact a set of equations in the components of **v**. So, multiplying the matrices gives us:

$$v_2 = 0$$

$$v_3 = 0$$

Suppose we let $v_1 = k$, and we can see that $v_2 = 0$ and $v_3 = 0$ as well. Thus, for any value of k, we have the eigenvector:

$$\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We go for the 'simplest form' of the eigenvector (the simplest non-zero multiple of this) to get:

eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 2$.